November 19th, 2009

MAT240: Abstract Linear Algebra Lecture:

Suppose $A \in M_{mxn}$ can be row-reduced to I. Find A^{-1}.

$$E_k \dots E_2 E_1 A = I \to A^{-1} = E_k \dots E_2 E_1$$

Turned practical the wrong way:

i. Row reduce A to I, write all elementary matrices you've encountered.
ii. Find A^{-1} by multiplying these elementary matrices in the opposite order.

$A^{-1} = A^{-1}I = E_k \dots (E_2(E_1 I))$

Moral:

A^{-1} is the result of doing to I the same row operations you did to A, to get to I.

Efficient Way:

$$(A|I) \xrightarrow{by\ row\ operations} (I|A^{-1})$$

Eg. $\begin{pmatrix} 1 & 0 & | & 1 & 0 \\ 0 & 1 & | & 0 & 1 \end{pmatrix}$

Eg. $\begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 3 & 4 & | & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - 3R_1} \begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 0 & -2 & | & -3 & 1 \end{pmatrix} \xrightarrow{-1/_2 R_2} \begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & 3/_2 & -1/_2 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & | & -2 & 0 \\ 0 & 1 & | & 3/_2 & -1/_2 \end{pmatrix}$

$$\to \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 0 \\ 3/_2 & -1/_2 \end{pmatrix}$$

How far can we go with row reduction?

Reduced Row Echelon Form:

1. Make the first non-zero entry in any row be 1 (the pivot).
2. In the column of every pivot, all else is 0.
3. Going down the rows, the pivots appear further and further to the right.

Eg.

$$\begin{pmatrix} 1 & 0 & 2 & 9 & 0 & 2 \\ 0 & 1 & 12 & -2 & 0 & e \\ 0 & 0 & 0 & 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, with column reduction, we can get rid of all entries save the pivots. Then by column interchanges, can get to $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$.

Theorem 1: The rank of an RREF matrix is the number of pivots in it = the number of non-zero rows.

Theorem 2: If A_{mxn} is invertible, its RREF is I → the $(A|I)$ algorithm always works.

Reason: A, B is the RREF of A.

A invertible → $R(A) = F^n \rightarrow rankA = n$

→ B has n pivots; each one is in a different column, there are n columns → 1 pivot in each row and in each column.