MAT240 — Nov. 16, 2006.

$$m\left\{\begin{array}{l} a_{11} + x_1 + \cdots \; a_{1n}x_n = b_1 \\ \vdots \\ \vdots \\ a_{m1}x_1 + \cdots \; a_{mn}x_n = b_m \end{array}\right\} \text{ system of linear equations}$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{pmatrix} 2 & -7 \\ -3 & 2 \end{pmatrix}$$

$$x - y = 7 \qquad x \longmapsto \begin{bmatrix} x \\ y \end{bmatrix} \qquad b \mapsto \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

$$3\cancel{y} + \cancel{x} = 2$$

$$x + 3y = 2 \qquad A = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \checkmark$$

• variables have to be in same order

$Ax = b \iff$ our system.

<u>Lucky case</u>: A is square and invertible

$A^{-1}Ax = A^{-1}b \iff$ our system

$Ix = A^{-1}b$

$x = A^{-1}b$ ✓ : it's solved.

<u>Theory</u>.

Case ① $\boxed{b=0}$ "homogeneous system"

<u>Claim</u>:

Consider $Ax = 0$.

1. $x = 0$ is always a solution.
2. x is a solution iff $x \in \text{nullspace}(A) = \ker(A)$

• $A \in M_{m\times n}(F)$ also a linear trans. $A: F^n \to F$

Case ② $\boxed{b \neq 0}$ "non-homogeneous system"

<u>Claim</u>: Consider $Ax = b$.

1. System has sol'n iff $b \in \text{range}(A) = \text{im}\, A$.
2. If x_0 solves the system ($Ax_0 = b$)

x_1 solves the system iff $x_1 = x_0 + x$ where x solves the homogenized version $Ax = 0$.

Pf. We know that $Ax_0 = b$

$$\cancel{Ax = b \Leftrightarrow x_1 = x_0 + x}$$

$$Ax_1 = b \Leftrightarrow A(x_0 + x) = b \Leftrightarrow Ax_0 + Ax = b \Leftrightarrow$$

$$x_1 = x_0 + x \qquad\qquad b + Ax = b \Leftrightarrow$$

$$Ax = 0$$

$\square$

<u>With row reduction.</u>

$$A \rightsquigarrow A' = \underset{\sim}{E}A \quad \text{"in r.r.e.f"}$$

E = a product of elementary matrices, hence invertible.

$$Ax = b \Leftrightarrow \overbrace{EA}^{A'}x = \underbrace{Eb}_{b'} \Leftrightarrow \underbrace{A'x = b'}_{\text{Easy to solve.}}$$

$$T(Ax) = T(b)$$

<u>Mechanical Aside.</u>: To find A' and b' at the same time, form

$$\tilde{A} = (A|b)$$

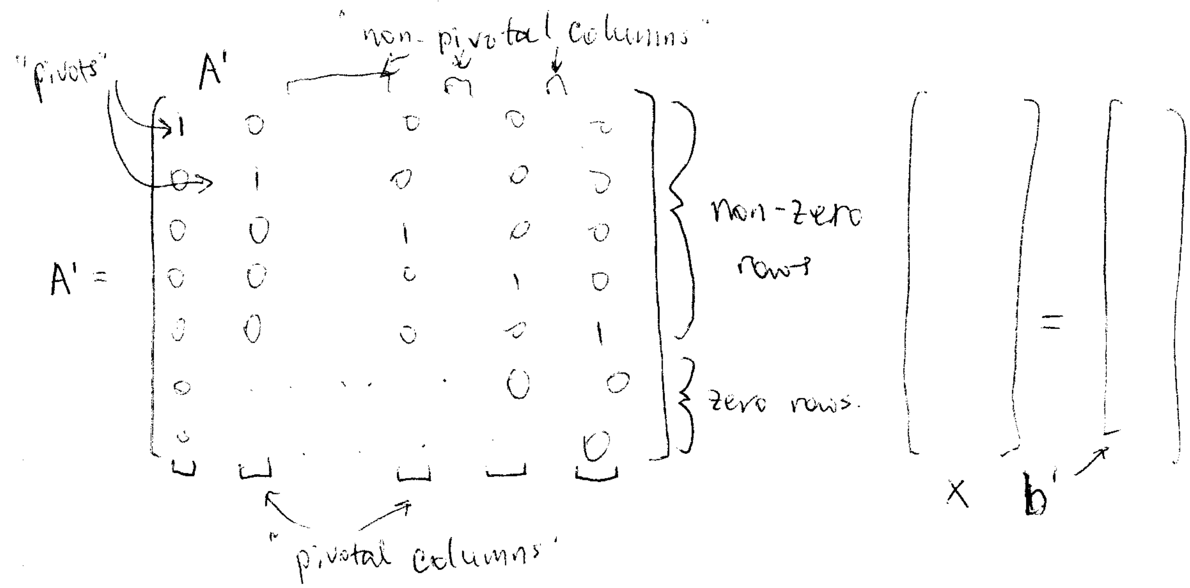
row reduce it

$$E\tilde{A} = (\underbrace{EA}_{A'} | \underbrace{Eb}_{b'})$$

eg. $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $b = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ $\quad \tilde{A} = \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 9/2 \\ 0 & 1 & 5/2 \end{array}\right]$

$$\qquad\qquad A' \quad b'$$

$$\therefore x = \begin{bmatrix} \frac{9}{2} \\ \frac{5}{2} \end{bmatrix}$$



Thm. 1. If b' has a non-zero entry in one of the zero rows of A', no solution.

2. Otherwise, all solutions are obtained by setting the entries of x corresponding to the non-pivotal columns in an arbitrary way, and then the rest is uniquely determined by solving single linear equations.