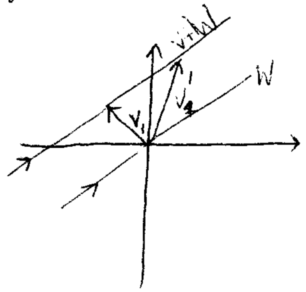
MAT240- Tutorial. Oct. 26, 2006.

Quotient Spaces.

given a vector space V and a subspace W.



$V = \mathbb{R}^2$

Define coset of a vector $v \in V$,

$$v + W = \{v + w \mid w \in W\}$$

Space of all cosets is denoted by V/W called V mod W.

Quotient Spaces

Last week, we showed that

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$
$$r(v + W) = (rv) + W$$

& V/W with above operations is a v.s

Property: 1. $u_1 + W = u_2 + W \Leftrightarrow u_1 - u_2 \in W$

2. $\dim(V/W) = \dim V - \dim W$ if $\dim V < \infty$

$f : V \to U$ linear transformation.

$\tilde{f} : ? \to ?$ Want to make it isomorphic.

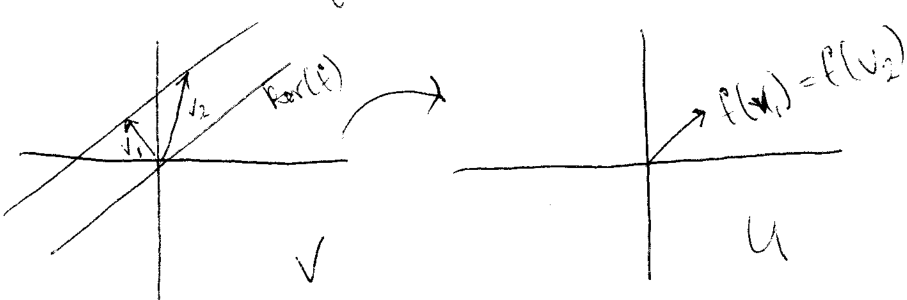
$\bar{f} : V \to R(f)$ and $\bar{f}(v) = f(v)$ [U was shrunk to its range]

$\tilde{f} : V/\ker(f) \to R(f)$

Define by $\tilde{f}(v + \ker(f)) = f(v)$

Is $\tilde{f}$ well-defined? i.e. if $v + \ker(f) = w + \ker(f)$ ①

then $f(v) = f(w)$



$$f(v_1 - v_2) = 0$$
$$f(v_1) = f(v_2)$$

① $\Leftrightarrow v-w \in Ker(f)$
$\Leftrightarrow f(v-w)=0$
$\Leftrightarrow f(v)-f(w)=0.$

$\tilde{f}$ is an isomorphism?

① $\tilde{f}(v+ker(f))=0$ Want $v+ker(f)=ker(f)$
$\Rightarrow f(v)=0$
$\Rightarrow v \in ker(f)$
$\Leftrightarrow v+ker(f)=ker(f)$ (by property 1)
$\therefore$ 1-1.

② $\tilde{f}$ is onto.
Pick a vector $f(v)$
Want to find $w+ker(f)$ s.t. $\tilde{f}(w+ker(f))=f(v)$
Just pick $w=v$.
$\therefore$ onto.

<u>Thm</u> $f: V \to U$ linear (First isomorphism thm).
$\exists \tilde{f}: V/_{Ker(f)} \to R(f)$

s.t. $\tilde{f}$ is an isomorphism

Exercise: Check $\tilde{f}$ is linear.

<u>Corollary</u>: (dimension formula)
$\dim V = \text{nullity}(f) + \text{rank}(f)$, if $\dim V < \infty$

Pf. $\dim\left(V/_{Ker(f)}\right) = \dim(R(f)) = \text{rank}(f)$
$\dim V - \text{nullity}(f)$
by property 2 □. $\approx$ Ex 40 in Section 2.1

Related to Quotients: P.23 #31, p.58 #35, p.79 #40

MAT240 - Tutorial

.6 #26 $P_n(\mathbb{R})$ $\quad \sum_{i=0}^{n} a_i x^i = g(x)$

$\cup$

$\{f \mid f(a) = 0\} = W \ni g$

$$0 = g(a) = \sum_{i=0}^{n} a_i a^i = a_0 + a_1 a + a_2 a^2 + \dots + a_n a^n$$

$$\Leftrightarrow a_0 = -\sum_{i=1}^{n} a_i a^i$$

$$g(x) = -\sum_{i=1}^{n} a_i a^i + \sum_{i=1}^{n} a_i x^i \in W$$

$$= \sum_{i=1}^{n} a_i (x^i - a^i)$$

$\therefore W = \operatorname{span}\{x^i - a^i \mid i = 1, \dots, n\}$

#28 V is a v.s. / $\mathbb{C}$ w/ dim n.

Show the real dim of V is $2n$

Let $\{v_1, \dots, v_n\}$ be a basis of V over $\mathbb{C}$.

$\Leftrightarrow$ any $v \in V$ has a unique representation as

$$v = \sum_{j=1}^{n} c_j v_j$$

$c_j = a_j + i b_j \qquad v = \sum_{j=1}^{n} a_j v_j + \sum_{j=1}^{n} b_j i v_j$

$\{v_1, \dots, v_n, iv_1, \dots, iv_n\}$ is a basis over $\mathbb{R}$.