Problem Set 13 — MAT257

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Disclaimer—This page has been typeset by a student as a *convenient* consolidation of the homework problems. There inevitably will be mistakes; always defer to the official handout!

Problems marked with $*$ are to be sumbitted for credit.

1 Munkres §25 (pp.217–218)

1. Check the computations made in Example 2 (Munkres, p.216).

2. Let $\alpha(t)$, $\beta(t)$, $f(t)$ be real-valued functions of class $\mathcal{C}^1$ on [0,1], with $f(t) > 0$. Suppose M is a 2-manifold in $\mathbb{R}^3$ whose intersection with the plane $z = t$ is the circle

$$\big(x - \alpha(t)\big)^2 + \big(y - \beta(t)\big)^2 = \big(f(t)\big)^2; \quad z = t$$

   if $0 \leq t \leq 1$, and is empty otherwise.

   (a) Set up an integral for the area of M. [*Hint:* Proceed as in Example 2 (Munkres, p.216).]

   (b) Evaluate when α and β are constant and $f(t) = (1+t)^{1/2}$.

   (c) What form does the integral take when f is constant and $\alpha(t) = 0$ and $\beta(t) = at$? (This integral cannot be evaluated in terms of the elementary functions.)

3. Consider the torus T of Exercise 7 of §17 (Munkres, p.151).

   (a) Find the area of this torus. [*Hint:* The cylindrical coordinate transformation carries a cylinder onto T. Parametrize the cylinder using the fact that its cross-section are circles.]

   (b) Find the area of that portion of T satisfying the condition $x^2 + y^2 \geq b^2$.

$*$ 4. Let M be a compact k-manifold in $\mathbb{R}^n$. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry; let $N = h(M)$. Let $f : N \to \mathbb{R}$ be a continuous function. Show that N is a k-manifold in $\mathbb{R}^n$, and

$$\int_N f \mathrm{d}V = \int_M (f \circ h) \mathrm{d}V.$$

   Conclude that M and N have the same volume.

5. (a) Express the volume of $S^n(a)$ in terms of the volume of $B^{n-1}(a)$. [*Hint:* Follow the pattern of Example 2 (Munkres, p.216).]

   (b) Show that for $t > 0$,

$$v\big(S^n(t)\big) = Dv\big(B^{n+1}(t)\big).$$

   [*Hint:* Use the result of Exercise 6 of §19 (Munkres, pp.168–169).]

6. The **centroid** of a compact manifold M in $\mathbb{R}^n$ is defined by a formula like that given in Exercise 3 of §22 (Munkres, p.193). Show that if M is symmetric with respect to the subspace $x_i = 0$ of $\mathbb{R}^n$, then $c_i(M) = 0$.

7. Let $E^n_+(a)$ denote the intersection of $S^n(a)$ with the upper half-space $\mathbb{H}^{n+1}$. Let $\lambda_n = v\big(B^n(1)\big)$.
   (a) Find the centroid of $E^n_+(a)$ in terms of λ_n and λ_{n-1}.
   (b) Find the centroid of $E^n_+(a)$ in terms of the centroid of $B^{n-1}_+(a)$. (See the exercises of §19 (Munkres, pp.168–169).)

∗ 8. Let M and N be compact manifolds without boundary in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.
   (a) Let $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ be continuous. Show that
   $$\int_{M\times N} f \cdot g \,\mathrm{d}V = [\int_M f \,\mathrm{d}V][\int_N g \,\mathrm{d}V].$$
   (b) Show that $v(M \times N) = v(M) \cdot v(N)$.
   (c) Find the area of the 2-manifold $S^1 \times S^1$ in $\mathbb{R}^4$.

2 Munkres §26 (p.226)

1. (a) Show that if $f, g : V^k \to \mathbb{R}$ are multilinear, so is $af + bg$.
   (b) Check that $\mathcal{L}^k(V)$ satisfies the axioms of a vector space.

2. (a) Show that if f and g are multilinear, so is $f \otimes g$.
   (b) Check the basic properties of the tensor product (Theorem 26.4).

3. Verify (2) and (3) of Theorem 26.5.

4. Determine which of the following are tensors on $\mathbb{R}^4$, and express those that are in terms of the elementary tensors on $\mathbb{R}^4$:
$$f(\mathbf{x}, \mathbf{y}) = 3x_1y_2 + 5x_2x_3,$$
$$g(\mathbf{x}, \mathbf{y}) = x_1y_2 + x_2y_4 + 1,$$
$$h(\mathbf{x}, \mathbf{y}) = x_1y_1 - 7x_2y_3.$$

∗ 5. Determine which of the following are tensors on $\mathbb{R}^4$, and express those that are in terms of the elementary tensors on $\mathbb{R}^4$:
$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 3x_1x_2z_3 - x_3y_1z_4,$$
$$g(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}) = 5x_3y_2z_3u_4v_4,$$
$$h(\mathbf{x}, \mathbf{y}, \mathbf{z}) = x_1y_2z_4 + 2x_1z_3.$$

∗ 6. Let f and g be the following tensors on $\mathbb{R}^4$:
$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 2x_1y_2z_2 - x_2y_3z_1,$$
$$g = \phi_{2,1} - 5\phi_{3,1}.$$
   (a) Express $f \otimes g$ as a linear combination of elementary 5-tensors.
   (b) Express $(f \otimes g)(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v})$ as a function.

7. Show that the four properties stated in Theorem 26.4 characterize the tensor product uniquely, for finite-dimensional spaces V.

8. Let f be a 1-tensor on $\mathbb{R}^n$; then $f(\mathbf{y}) = A \cdot \mathbf{y}$ for some matrix A of size $1 \times n$. If $T : \mathbb{R}^m \to \mathbb{R}^n$ is the linear transformation $T(\mathbf{x}) = B \cdot \mathbf{x}$, what is the matrix of the 1-tensor T^*f on $\mathbb{R}^m$?

3 "In addition..."

A. Let $\alpha : \{(u,v) \in \mathbb{R}^2 \colon u^2+v^2 \leq 1\} \to \mathbb{R}^3$ be given by $\alpha(u,v) = \big(u-v,\, u+v,\, 2(u^2+v^2)\big)$. Let M be the image of α.

(a) Describe M.

(b) Show that M is a manifold.

(c) Find the boundary ∂M of M.

∗ (d) Find the volume $V(M)$ of M.

∗ (e) Find $\int_M z\, \mathrm{d}V$.