Jan. 6th. **Agenda** 1. Ruin a perfectly clean blackboard

2. Get back to the 257 atmosphere

3. Volumes, isometrics.

4. Nothing about TT2.

Read Along. Reread 16-20 esp. 20.

Riddle Along Is there a distance-reducing $F:[0,1]^2 \to \mathbb{R}^2$, meaning $d(f(x),f(y)) \leq d(x,y)$; s.t. $\text{length}(Bd(f[0,1]^2)) > 4$

Thm. Given $\underset{\subset \mathbb{R}^n}{A} \xrightarrow{g} \underset{\subset \mathbb{R}^n}{B} \xrightarrow{f} \mathbb{R}$, $\int_B f = \int_A \underbrace{(f\circ g)}_{\text{"the pullback of } f\text{"}} |\det Dg|$

Corollary Let $P(v_1,\cdots,v_n)$ be the Parallelopiped spanned by $v_1,\cdots,v_n$

$P(v_1,\cdots,v_n) = \{\sum a_i v_i : 0 \leq a_i \leq 1\}$

Then $\text{Vol}(P(v_1,\cdots,v_n)) = |\det(v_1|v_2|\cdots|v_n)|$

"geometric intepretation of det"

Pf. $\underset{[0,1]^n}{A} \xrightarrow{g} \underset{P(v_1,\cdots,v_n)}{B} \xrightarrow{f=1} \mathbb{R}$

$g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \sum x_i v_i = M\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

clearly $P(v_1,\cdots,v_n) = g([0,1]^n)$ where $M = (v_1|v_2|\cdots|v_n)$.

By CVT, $\int_B 1 = \int_{P(v_1,\cdots,v_n)} 1 = \text{Vol}(P(v_1,\cdots,v_n))$.

$\parallel \int_A 1\cdot|\det Dg| = \int_{[0,1]^n} 1\cdot|\det(v_1|v_2|\cdots|v_n)| = 1\cdot|\det(v_1|v_2|\cdots|v_n)|$

Exercise Compute the volume of

$B^3 = \{x \in \mathbb{R}^3 : \|x\| < 1\}$ $\|x\| = (\sum x_i^2)^{1/2}$.

$= [(x_1,\cdots,x_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}]^{1/2} = (x^T \cdot x)^{1/2}$

θ = longitude (east-west) $\in [-\pi,\pi]$

ϕ = latitude (South-North) $\in [-\pi/2, \pi/2]$

r: radius $\in [0,1]$

$g: [0,1]\times[-\pi,\pi]\times[-\pi/2,\pi/2] \to B^3$

where $g\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} r\cos\phi\cos\theta \\ r\cos\phi\sin\theta \\ r\sin\phi \end{pmatrix}$ $\qquad \|g\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}\|^2 = r^2\cos^2\phi\cos^2\theta + r^2\cos^2\phi\sin^2\theta + r^2\sin^2\phi = r^2$

$$Dg = \begin{pmatrix} \cos\phi\cos\theta & -r\cos\phi\sin\theta & -r\sin\phi\cos\theta \\ \cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi & 0 & r\cos\phi \end{pmatrix}$$

$$|\det Dg| = |r^2\cos\phi|$$

$$\mathrm{Vol}(B^3) = \int_{B^3} 1 \overset{CVT}{=} \int_{[0,1]\times[-\pi,\pi]\times[-\pi/2,\pi/2]} r^2\cos\phi$$

Fub

$$\int_0^1 dr \int_{-\pi}^{\pi} d\theta \int_{-\pi/2}^{\pi/2} r^2\cos\phi \, d\phi = \int_0^1 dr \int_{-\pi}^{\pi} d\theta \, 2r^2$$

$$= \int_0^1 dr \, 4\pi r^2 = \left[\tfrac{4}{3}\pi r^3\right]_0^1$$

Def $h: \mathbb{R}^n \to \mathbb{R}^n$ is called an "isometry"

if $\forall x,y \quad d(h(x), h(y)) = d(x,y)$

Thm $h: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry

iff it can be written in the form $h(x) = P + Ax$

where $P \in \mathbb{R}^n$ and $A \in M_{n\times n}$ s.t. $A^T \cdot A = Id$

comments: isometries are "volume preserving"

Pf By CVT, $\det(Dh) = \det(A)$

$$\det(A^T\cdot A) = \det(A^T)\cdot\det(A) = \det(A)^2$$

$$\Rightarrow \det(A) = \pm 1$$