Section 11.

2. Thm: no non-trivial open set in $\mathbb{R}^n$ has measure zero in $\mathbb{R}^n$

proof: Let A be a non-trivial open set in $\mathbb{R}^n$ and $a \in A$ (as A non-trivial).
By the def. of open sets, $\exists r>0$ s.t $B(a,r) \subseteq A$ where $B(a,r) = \{x \in \mathbb{R}^n : d(a,x) < r\}$.
Let $\varepsilon = \frac{1}{2} V(B(a,r))$, now we have: they must cover $B(a,r)$ i.e. $\bigcup_{i=1}^{\infty} Q_i \supseteq B(a,r)$.
for any cover of A, say $Q_1, Q_2, \ldots$ rectangles. We have $\sum_{i=1}^{\infty} V(Q_i) \geqslant V(\bigcup_{i=1}^{\infty} Q_i) \geqslant V(B(a,r)) > \varepsilon$.
By the def. of measure-0, A is not measure-0. □

4. Thm: set of irrationals in $[0,1]$ does not have measure zero in $\mathbb{R}$.

proof: Since there are uncountably many reals in $[0,1]$ and countably many rationals in $[0,1]$
$\Rightarrow$ there are uncountably many irrationals in $[0,1]$. Let A = set of irrationals in $[0,1]$.
Let $Q_1, Q_2, \ldots$ be countably many rectangles in $\mathbb{R}$ that covering A.
But A has uncountably many elements. therefore $\exists$ at least one Q_i has uncountably elements of A (if Q_j has countably many elements of A $\forall j \in \mathbb{N}$, then $\bigcup_{j=1}^{\infty} Q_j$ has countably many elements of A)
WLOG Let Q_1 be the rectangle containing uncountably elements of A.
Let $A_1 \subseteq A$ be the set of elements contained by Q_1
Thus $\sup Q_1 \geqslant \sup A_1 > \inf A_1 \geqslant \inf Q_1$ i.e. $\sup Q_1 - \inf Q_1 > 0$.
Let $\varepsilon = \frac{1}{2}(\sup Q_1 - \inf Q_1)$ because A_1 has >1 elements, so $\sup A_1 \neq \inf A_1$
Then $\sum_{i=1}^{\infty} V(Q_i) \geqslant V(Q_1) \geqslant (\sup Q_1 - \inf Q_1) > \varepsilon$, holds for any cover of A. □

6. $f: [a,b] \to \mathbb{R}$ $G_f = \{(x,y) \mid y = f(x)\}$ Thm: f is continuous $\Rightarrow$ G_f has measure 0 in $\mathbb{R}^2$

10

proof: f is cont. on $[a,b]$ $\Rightarrow$ f is unif. cont. on $[a,b]$. Let $\varepsilon > 0$ be given
i.e. $\exists \delta > 0$, $\forall x \in [a,b]$. $\forall \varepsilon_1 > 0$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon_1$. set $\varepsilon_1 = \frac{\varepsilon}{3(b-a)}$
Let $P = (a = t_0 < t_1 < \cdots < t_n = b)$ be a partition of $[a,b]$ where $\frac{\delta}{3} \leq t_{i+1} - t_i \leq \frac{\delta}{2}$ $0 \leq i \leq n-1$.
Let $R_i = [t_i, t_{i+1}]$ for $0 \leq i \leq n-1$, then $\forall x,y \in R_i \Rightarrow |x-y| \leq \frac{\delta}{2} < \delta \Rightarrow |f(x) - f(y)| < \varepsilon_1$
that is $|\sup_{x \in R_i} f(x) - \inf_{x \in R_i} f(x)| = \sup_{x \in R_i} f(x) - \inf_{x \in R_i} f(x) < \varepsilon_1$
(because R_i is a closed interval in $[a,b]$ and f is continuous, therefore $\sup_{x \in R_i} f(x) = f(x_1)$
and $\inf_{x \in R_i} f(x) = f(x_2)$ for some $x_1, x_2 \in R_i$. therefore $|\sup_{x \in R_i} f(x) - \inf_{x \in R_i} f(x)| = |f(x_1) - f(x_2)| < \varepsilon_1$)
Set $Q_i = [t_{i-1}, t_i] \times [\inf_{x \in R_{i-1}} f(x), \sup_{x \in R_{i-1}} f(x)]$ then $\bigcup_{i=1}^{n} Q_i$ covers G_f.
And we have $\sum_{i=1}^{\infty} V(Q_i) = \sum_{i=1}^{n} V(Q_i) = \sum_{i=1}^{n} (t_i - t_{i-1}) \times (\sup_{x \in R_{i-1}} f(x) - \inf_{x \in R_{i-1}} f(x))$
$< \sum_{i=1}^{n} \frac{\delta}{2} \times \varepsilon_1 = \frac{n\delta\varepsilon_1}{2}$
But $n \cdot \frac{\delta}{2} \leq \frac{3}{2}(n \cdot \frac{\delta}{3}) \leq \frac{3}{2}(t_n - t_{n-1} + t_{n-1} - t_{n-2} + \cdots + t_1 - t_0) = \frac{3}{2}(t_n - t_0) = \frac{3}{2}(b-a)$.
$\Rightarrow \sum_{i=1}^{\infty} V(Q_i) < \frac{n\delta\varepsilon_1}{2} \leq \frac{3}{2}(b-a)\varepsilon_1 = \frac{3}{2}(b-a) \cdot \frac{\varepsilon}{3(b-a)} = \frac{\varepsilon}{2} < \varepsilon$
By def. of measure 0, G_f has measure 0 in $\mathbb{R}^2$ □