Homework 6. NOT HAND IN.

Section 10.

4. Let P be a partition of h on $[0,1]^2$ s.t. $P = (P_1, P_2)$ where $P_1 = (0, \frac{1}{n}, \frac{2}{n}, \dots, 1) = P_2$ where $n = \left\lceil \frac{2f(1)g(1)}{\epsilon} \right\rceil$ s.t. $n \in \mathbb{Z}^+$, $\epsilon > 0$

So. $L(h,P) = \sum_{R \in P} V(R) m_R(h) = n^2 \cdot (\frac{1}{n})^2 m_R(h)$.

$U(h,f) = \sum_{R \in P} V(R) M_R(h) \leq n^2 \cdot (\frac{1}{n})^2 M_R(h)$

$U(h,P) - L(h,P) = \sum_{R \in P} V(R)[M_R(h) - m_R(h)] = \sum_{i=1}^{n} \sum_{j=1}^{n} (\frac{1}{n})^2 [h(\frac{i}{n}, \frac{j}{n}) - h(\frac{i-1}{n}, \frac{j-1}{n})]$

since f, g are increasing & $h(x,y) = f(x) g(y)$.

So $M_R(h) - m_R(h) = [h(\frac{i}{n}, \frac{j}{n}) - h(\frac{i-1}{n}, \frac{j-1}{n})]$ where $i, j = 1, 2, 3, \dots, n$

so. $U(h,P) - L(h,P) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (f(\frac{i}{n}) g(\frac{j}{n}) - f(\frac{i-1}{n}) g(\frac{j-1}{n})) = \frac{1}{n^2} \sum_{j=1}^{n} [g(\frac{j}{n}) \sum_{i=1}^{n} f(\frac{i}{n}) - g(\frac{j-1}{n}) \sum_{i=1}^{n} f(\frac{i-1}{n})]$

$= \frac{1}{n^2} \sum_{j=1}^{n} [(g(\frac{j}{n}) - g(\frac{j-1}{n}))(f(\frac{1}{n}) + \dots + f(\frac{n-1}{n})) + g(\frac{j}{n}) f(1) - g(\frac{j-1}{n}) f(0)]$

$= \frac{1}{n^2} [(f(\frac{1}{n}) + \dots + f(\frac{n-1}{n}))(g(1) - g(0)) + (f(1) - f(0))(g(\frac{1}{n}) + \dots + g(\frac{n-1}{n})) + g(1) f(1) - g(0) f(0)]$

Since $f, g \geq 0$, so. $g(0) \geq 0$, $f(0) \geq 0$ $\leq \frac{1}{n^2} [(f(\frac{1}{n}) + \dots + f(\frac{n-1}{n})) g(1) + f(1)(g(\frac{1}{n}) + \dots + g(\frac{n-1}{n})) + f(1) g(1)$

Since $f(\frac{1}{n}) < \dots < f(1)$, $g(\frac{1}{n}) < \dots < g(1)$ $\leq \frac{1}{n^2} [(n-1) f(1) g(1) + f(1)(n-1) g(1) + f(1) g(1)]$

$= \frac{1}{n^2} (2n-1) f(1) g(1)$

$< \frac{1}{n^2} (2n) f(1) g(1) = \frac{2}{n} f(1) g(1)$

Since $n \in \mathbb{Z}^+$,

Since $n = \left\lceil \frac{2f(1)g(1)}{\epsilon} \right\rceil$, so $n \geq \frac{2f(1)g(1)}{\epsilon} \Rightarrow \frac{1}{n} \leq \frac{\epsilon}{2f(1)g(1)}$

If $f(1)g(1) \neq 0$ i.e. $f \neq 0$ & $g \neq 0$. So $U(h,P) - L(h,P) < \frac{1}{n} 2f(1)g(1) \leq \epsilon$.

Then h is integrable over $[0,1]^2$.

If $f = 0$ & $g = 0$ so $h = 0$, h is constant. then h is integrable on $[0,1]^2$.

Problem B.

Since f is unbounded & cont. on $(0,1]$, so. $f \to \pm\infty$ as $x \to 0$.

if $f(x) \to \infty$ as $x \to 0$ then given ϵ, whatever how small δ is, we still can find an interval $[x,y]$ near 0. s.t. $|y-x| < \delta$ but $|f(y) - f(x)| > \epsilon$.

since the function f goes so fast near 0.

we can apply similarly as $x \to 0$ $f(x) \to -\infty$.