$$A \xrightarrow[r_i *= c]{r_i \mapsto c r_i} E^2_{i,c} A$$

$$A \xrightarrow[c_i *= c]{c_i \mapsto c\, c_i} A \cdot E^2_{i,c}.$$

$$E^2_{i,c^{-1}} \cdot E^2_{i,c} = I$$

$E^3_{ij,c}$, $i \neq j$ c anything

$$\begin{array}{c c} & \begin{array}{cccc} i & & & j \end{array} \\ i & \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & 1 & c \\ & & & 1 \end{pmatrix} \end{array}$$

$E^3_{ij,c} \cdot A$ — A w/ c times row j added to row $\#i$.

$$\begin{array}{c c} & \begin{array}{cc} & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_i \\ r_j \end{pmatrix} \end{array} \begin{pmatrix} r_i + c r_j \\ r_j \end{pmatrix}$$

$$A \xrightarrow[r_i += c \cdot r_j]{r_i + c r_j \to r_i} E^3_{ij,c} A$$

$$A \xrightarrow[c_j += c \cdot c_i]{c_j + c c_i \to c_j} A E^3_{ij,c}$$

$$E^3_{ij,-c} E^3_{ij,c} = I.$$

m

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1. The following ops do not change rank

$A \xrightarrow{r_i \leftrightarrow r_j} E^1_{i,j} A \quad A \xrightarrow{c_i \leftrightarrow c_j} A E^1_{i,j}$

$c \neq 0 \quad A \xrightarrow{r_i *= c} E^2_{i,c} A \quad A \xrightarrow{c_i *= c} A E^2_{i,c}$

$A \xrightarrow{r_i += c r_j} E^3_{ijc} A \quad A \xrightarrow{c_j += c c_i} A E^3_{ijc}$

2. $\operatorname{rank}\left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right) = k$

Then you can always win the game.

Every matrix A can be brought to the form $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$ using row & col ops of the type on the left.

Example compute the rank of

$$A = \begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 4 & 4 & 4 & 8 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix} \xrightarrow{r_1 *= \frac{1}{4}} \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4 += -6r_1]{r_3 += -8r_1} \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix} \xrightarrow[\substack{r_3 *= -\frac{1}{2} \\ r_4 *= -1}]{r_2 *= \frac{1}{2}} \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 3 & 4 & 3 & -1 \\ 0 & 3 & 4 & 3 & -1 \end{pmatrix}$$

$$\xrightarrow[r_4 += -r_3]{\text{opportunistic move}} \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 3 & 4 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[r_3 += -3r_2]{r_1 += -r_2} \begin{pmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & -2 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_3 *= -\frac{1}{2}} \begin{pmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[r_2 += -2r_3]{r_1 += r_3} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$