Problem Set 14 — MAT257

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Problems marked with $*$ are to be submitted for credit.

1 Munkres §27 (p.236)

$*$ 1. Which of the following are alternating tensors on $\mathbb{R}^4$?

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}) &= x_1 y_2 - x_2 y_1 + x_1 y_1, \\ g(\mathbf{x}, \mathbf{y}) &= x_1 y_3 - x_3 y_2, \\ h(\mathbf{x}, \mathbf{y}) &= (x_1)^3 (y_2)^3 - (x_2)^3 (y_1)^3. \end{aligned}$$

2. Let $\sigma \in S_5$ be the permutation such that

$$\big(\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)\big) = (3, 1, 4, 5, 2).$$

Use the procedure given in the proof of Lemma 27.1 to write σ as a composite of elementary permutations.

3. Let ψ_I be an elementary alternating k-tensor on V corresponding to the basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$ for V. If $j_1, \ldots, j_k$ is an arbitrary k-tuple of integers from the set $\{1, \ldots, n\}$, what is the value of

$$\psi_I(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_k})?$$

4. Show that if $T : V \to W$ is a linear transformation and if $f \in \mathcal{A}^k(W)$, then $T^* f \in \mathcal{A}^k(V)$.

5. Show that

$$\psi_I = \sum_\sigma (\operatorname{sgn} \sigma) \phi_{I_\sigma},$$

where if $I = (i_1, \ldots, i_k)$, we let $I_\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(k)})$.

Hint: Show first that $(\phi_{I_\sigma})^\sigma = \phi_I$.

2 Munkres §28 (pp.243–244)

∗ 1. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^5$. Let

$$F(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 2x_2y_2z_1 + x_1y_5z_4,$$
$$G(\mathbf{x}, \mathbf{y}) = x_1y_3 + x_3y_1,$$
$$h(\mathbf{w}) = w_1 - 2w_3.$$

(a) Write AF and AG in terms of elementary alternating tensors.
Hint: Write F and G in terms of elementary tensors and use Step 9 of the proof of Theorem 28.1 to compute A_{ϕ_I}.

(b) Express $(AF) \wedge h$ in terms of elementary alternating tensors.

(c) Express $(AF)(\mathbf{x}, \mathbf{y}, \mathbf{z})$ as a function.

2. If G is symmetric, show that $AG = 0$. Does the converse hold?

3. Show that if $f_1, \dots, f_k$ are alternating tensors of orders $l_1, \dots, l_k$, respectively, then

$$\frac{1}{l_1! \cdots l_k!} A(f_1 \otimes \cdots \otimes f_k) = f_1 \wedge \cdots \wedge f_k.$$

4. Let $\mathbf{x}_1, \dots, \mathbf{x}_k$ be vectors in $\mathbb{R}^n$; let X be the matrix $X = [\mathbf{x}_1 \cdots \mathbf{x}_k]$. If $I = (i_1, \dots, i_k)$ is an *arbitrary* k-tuple from the set $\{1, \dots, n\}$, show that

$$\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}(\mathbf{x}_1, \dots, \mathbf{x}_k) = \det X_I.$$

5. Verify that $T^*(F^\sigma) = (T^*F)^\sigma$.

∗ 6. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be the linear transformation $T(\mathbf{x}) = B \cdot \mathbf{x}$.

(a) If ψ_I is an elementary alternating k-tensor on $\mathbb{R}^n$, then $T^*\psi_I$ has the form

$$T^*\psi_I = \sum_{[J]} c_J \psi_J,$$

where the ψ_J are the elementary alternating k-tensors on $\mathbb{R}^m$. What are the coefficients C_J?

(b) If $f = \sum_{[J]} d_I \psi_I$ is an alternating k-tensor on $\mathbb{R}^n$, express T^*f in terms of the elementary alternating k-tensors on $\mathbb{R}^m$.

3 "In addition..."

∗ A. A symmetric tensor is a tensor whose values are unchanged if its arguments are permuted. Along the lines of our development of a theory of "tensors" and a theory of "alternating tensors", develop a theory of "symmetric tensors" $S^k(V)$. Your theory should include:

1 definitions for specific tensors σ_I for $I \in \underline{n}_s^k$ (what should $\underline{n}_s^k$ be?),

2 a proof that the σ_I exist, are unique, and form a basis of $S^k(V)$, and

3 a computation of the dimension of $S^k(V)$.

∗ B. Find a good way of identifying $A^1(\mathbb{R}^3)$ and $A^2(\mathbb{R}^3)$ with $\mathbb{R}^3$. Under that identification, $\wedge : A^1(\mathbb{R}^3) \times A^1(\mathbb{R}^3) \to A^2(\mathbb{R}^3)$ becomes a map $P : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. If you chose your identifications right, P is the vector product of two vectors in $\mathbb{R}^3$. See to it that this is the indeed the case!

C. The determinant, as a function of a list of column vectors, is alternating. Write it in terms of the elementary alternating functions ψ_I.