Problem 3. Let G be a finite group, let p be a prime number, let α be the largest natural number such that $p^\alpha||G|$, and let P be a subgroup of G whose order is p^α.

1. Suppose that $x \in G$ is an element whose order is a power of p, and suppose that x normalizes P. Show that $x \in P$.

If x normalizes P then $x \in N_G(P)$ and $\langle x \rangle \prec N_G(P)$ because $< N_G(P)$ is a group and has closure. Now we know that $\langle x \rangle, P \prec G$ and $\langle x \rangle \prec N_G(P)$. So by the 2nd isomorphism theorem $P\langle x \rangle \prec G$, $P \lhd P\langle x \rangle$, $P \cap \langle x \rangle \lhd \langle x \rangle$, and $\frac{|P\langle x \rangle|}{|P|} = \frac{|\langle x \rangle|}{|P \cap \langle x \rangle|}$.

We know that $|P| = p^\alpha$ from the definition of Sylow-p groups and $|\langle x \rangle| = p^\alpha$ $(1 \leq \beta \leq \alpha)$ by the definition of $|x|$. Now assume that $x \notin P$. This means that $|P \cap \langle x \rangle| = 1$ (the identity). And if we substitute in the values to $\frac{|P\langle x \rangle|}{|P|} = \frac{|\langle x \rangle|}{|P \cap \langle x \rangle|}$ we get $\frac{|P\langle x \rangle|}{p^\alpha} = \frac{p^\beta}{1} \to |P\langle x \rangle| = p^{\alpha+\beta}$.

This is a contradiction because $P\langle x \rangle$ is a subgroup of G. Since it is a subgroup of G and $P^\alpha < P^{\alpha+\beta}$ we see that its order can't divide $|G|$ because from the definition of Sylow subgroups, α is the highest power of p that divides g. From Lagrange's theorem, the order of every subgroup must divide the order of the group which gives us the contradiction. Therefore $x \in P$.$\square$