PSet 14: Partial Solutions

DISCLAIMER: I cannot claim that what I have written here constitutes a perfect solution. Certainly some mistakes are present; hopefully these mistakes aren't too severe. I hope that my answers may serve as a guide to you when studying for the final exam.

Problem 27.1.

Part i. We claim that f is not an alternating tensor in $\mathbb{R}^4$. Take $x = y = e_1 \in \mathbb{R}^4$. Then $f(x,y) = f(e_1,e_1) = 0+0+1$, whereas $f(y,x) = f(e_1,e_1) = 0+0+1 \neq -f(x,y)$. Hence, by definition, f must not an alternating tensor.

Part ii. We claim that g is not an alternating tensor in $\mathbb{R}^4$. Take $x = e_1$ and $y = e_3$, both in $\mathbb{R}^4$. Then $g(x,y) = g(e_1,e_3) = 1-0$, whereas $g(y,x) = g(e_3,e_1) = 0-0 \neq -g(x,y)$. Hence, by definition, g must not an alternating tensor.

Part iii. h is not an alternating tensor in $\mathbb{R}^4$ since it is not a tensor in $\mathbb{R}^4$. We show that h fails to be linear in its first coordinate: take $x = \widetilde{x} = e_1$, and $y = e_2$. Then $h(x+\widetilde{x},y) = 8\cdot 1 + 0$, but $h(x,y) + h(\widetilde{x},y) = 1+1 \neq h(x+\widetilde{x},y)$.

Problem 28.1.

Part a. We first write F and G in terms of elementary tensors. $F(x,y,z) = 2x_2y_2z_1 + x_1y_5z_4 = (2\phi_{2,2,1} + \phi_{1,5,4})(x,y,z) \implies F = 2\phi_{2,2,1} + \phi_{1,5,4}$, where the ϕ terms are elementary 3-tensors corresponding to the standard basis for $\mathbb{R}^5$. Similarly, $G(x,y) = (\phi_{1,3} + \phi_{3,1})(x,y) \implies G = \phi_{1,3} + \phi_{3,1}$, where the ϕ terms are elementary 2-tensors corresponding to the standard basis for $\mathbb{R}^5$. Hence,

$$\begin{aligned}
AF &= A(2\phi_{2,2,1} + \phi_{1,5,4}) \\
&= A(2\phi_2 \otimes \phi_2 \otimes \phi_1 + \phi_1 \otimes \phi_5 \otimes \phi_4) \\
&= 2A(\phi_2 \otimes \phi_2 \otimes \phi_1) + A(\phi_1 \otimes \phi_5 \otimes \phi_4) && \text{by linearity,} \\
&= 2\phi_2 \wedge \phi_2 \wedge \phi_1 + \phi_1 \wedge \phi_5 \wedge \phi_4 && \text{by step 9 of Theorem 28.1,} \\
&= \phi_1 \wedge \phi_5 \wedge \phi_4 && \phi_2 \wedge \phi_2 = 0 \text{ as } \phi_2 \text{ is alternating of odd order,} \\
&= -\psi_{1,4,5}
\end{aligned}$$

And:

$$\begin{aligned}
AG &= A(\phi_{1,3}) + A(\phi_{3,1}) && \text{by linearity} \\
&= A(\phi_1 \otimes \phi_3) + A(\phi_3 \otimes \phi_1) \\
&= \phi_1 \wedge \phi_3 + \phi_3 \wedge \phi_1 && \text{by step 9 of Theorem 28.1,} \\
&= \phi_1 \wedge \phi_3 + (-1)^{1\cdot 1}\phi_1 \wedge \phi_3 \\
&= 0
\end{aligned}$$

Part b. We note that $h(w) = w_1 - 2w_3 = (\phi_1 - 2\phi_3)(w) \implies h = \phi_1 - 2\phi_3$. Then,

$$\begin{aligned}
(AF) \wedge h &= (\phi_1 \wedge \phi_5 \wedge \phi_4) \wedge (\phi_1 - 2\phi_3), \\
&= \phi_1 \wedge \phi_5 \wedge \phi_4 \wedge \phi_1 - 2\phi_1 \wedge \phi_5 \wedge \phi_4 \wedge \phi_3 && \text{by distributivity,} \\
&= (-1)^2\phi_5 \wedge \phi_4 \wedge (\phi_1 \wedge \phi_1) - 2\phi_1 \wedge \phi_5 \wedge \phi_4 \wedge \phi_3 && \text{by associativity and anticommutativity,} \\
&= -2\phi_1 \wedge \phi_5 \wedge \phi_4 \wedge \phi_3 \\
&= 2\psi_{1,3,4,5}
\end{aligned}$$

Part c. We compute,

$$\begin{aligned}(AF)(x,y,z) &= (\phi_1 \wedge \phi_5 \wedge \phi_4)(x,y,z) \\ &= \psi_{1,5,4}(x,y,z) && \text{by Theorem 28.1 (5),} \\ &= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_5 & y_5 & z_5 \\ x_4 & y_4 & z_4 \end{vmatrix} && \text{by Theorem 27.7,} \\ &= x_1(y_5 z_4 - z_5 y_4) - y_1(x_5 z_4 - x_4 z_5) + z_1(x_5 y_4 - y_5 x_4).\end{aligned}$$

Problem 28.6. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be the linear transformation $T(x) = Bx$.

Part a. Let us assume that the alternating tensors ψ_J on $\mathbb{R}^m$ are generated with respect the standard basis for $\mathbb{R}^m$, i.e. $\{e_1, \dots, e_m\}$. If $I' = (i'_1, \dots, i'_k)$, then $T^*\psi_I(e_{I'}) = \psi_I(T(e_{i'_1}), \dots, T(e_{i'_k})) = \psi_I(B_{i'_1}, \dots, B_{i'_k})$ where B_i is the i-th column of the matrix B. Let $B_{I'} := \left(B_{i'_1} | \dots | B_{i'_k}\right)$, i.e. the matrix obtained by taking the columns I' of B. Now, by Theorem 27.7, we may evaluate $\psi_I(B_{I'})$ as a determinant:

$$\psi_I(B_{I'}) = \det(B^I_{I'})$$

where $B^I_{I'}$ is the square matrix obtained by taking the rows I of the matrix $B_{I'}$. Finally, express $T^*\psi_I$ as:

$$T^*\psi_I = \sum_{[J]} c_J \psi_J.$$

Then,

$$\begin{aligned}T^*\psi_I(e_{I'}) &= \sum_{[J]} c_J \psi_J(e_{I'}) \\ &= c_{I'} && \text{as } \psi_J(e_I) = \delta_{IJ}, \\ \implies \det(B^I_{I'}) &= c_{I'}\end{aligned}$$

Part b. Let $f = \sum_{[I]} d_I \psi_I$ be an alternating k-tensor on $\mathbb{R}^n$. Then,

$$\begin{aligned}T^*f &= \sum_{[I]} d_I T^*\psi_I && \text{by linearity} \\ &= \sum_{[I]} d_I \left(\sum_{[J]} \det(B^I_J) \right) \psi_J && \text{by part a., and in the notation of part a.}\end{aligned}$$

Note also that if $I' = (i'_1, \dots, i'_k)$, then $f(e_{I'}) = \sum_{[I]} d_I \psi_I(e_{I'}) = d_{I'}$. Hence,

$$T^*f = \sum_{[I]} f(e_I) \left(\sum_{[J]} \det(B^I_J) \right) \psi_J = \sum_{[I]} \sum_{[J]} f(e_I) \det(B^I_J) \psi_J.$$