

Example In \mathbb{R}^3 find eqn. of the plane through

A(3, 6, -7), B(-2, 0, 4), C(5, -9, -2)

plug these numbers into ④

$$x = (x_1, x_2, x_3) = (3, 6, -7) + s(-5, -6, 11) + t(2, -15, 5) \quad s, t \in \mathbb{R}$$

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$$W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1 = 3a_2, a_3 = -a_2\}$$

$$W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid 2a_1 - 7a_2 + a_3 = 0\} \quad Q = W_1 \cap W_3$$

$$W_1 \cap W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid \begin{cases} a_1 - 3a_2 = 0 \\ a_3 + a_2 = 0 \\ 2a_1 - 7a_2 + a_3 = 0 \end{cases}\}$$

26/09/06 (lecture #5)

Vector Space (cont')

anything.

Examples ① $\{0\}$ is a v.s. over $\mathbb{F} = \{0, 1, a, b\}$

② \mathbb{F}^n (a_1, \dots, a_n) but we write this instead

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \text{ i.e. } \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

③ $M_{m \times n}(\mathbb{F})$ we have $a_{m \times n} \in M_{m \times n}$ $\left\{ \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \right\}$. If $[M_{m \times 1}] (\mathbb{F}) = \mathbb{F}^m$

④ $\mathbb{C}/\mathbb{R}, \mathbb{R}/\mathbb{Q}$ (if \mathbb{R} is a subset of \mathbb{C} then \mathbb{C} is a vector space of \mathbb{R}).

⑤ $\mathcal{F}(S, F)$ collection of all function from Set S to function F is a vector space \mathcal{F} .

5. Polynomials

$$7x^3 + 9x^2 - 2x + 17$$

Let F be a field

$$P(F) = \left\{ \sum_{i=0}^n a_i x^i : \begin{matrix} n \in \mathbb{Z}, n \geq 0 \\ \forall i \quad a_i \in F \end{matrix} \right\}$$

Addition of polynomials is defined in the expected way

$$\sum_{i=0}^n a_i x^i + \sum_{i=0}^m b_i x^i = \sum_{i=0}^{\max(n,m)} (a_i + b_i) x^i$$

Thm 1 (Cancellation law for v.s.)

If z is a v.s.

$$x+z = y+z$$

then $x=y$

Prf: add w to both sides of the given eqn. where w is an element

for which $z+w=0$ (exists by v.s. 4)

$$(x+z)+w = (y+z)+w$$

$$x+(z+w) = y+(z+w) \quad \text{by v.s. 2}$$

$$x+0 = y+0 \quad \text{by choice of } w$$

$$x=y \quad \blacksquare \quad \text{by v.s. 3}$$

Thm 2

" 0 is unique" ... If some $z \in V$

satisfies $x+z=x$... for some $x \in V$. (even if for one $x \rightarrow$ stronger statement)

then $z=0$

Prf: $x+z=x+0$... $\exists y$ s.t. $x+y=0$ add. invers.

$$z+x=0+x \quad (z+x)+y=(0+x)+y$$

$$z=0 \quad z+(x+y)=0+(x+y) \Rightarrow z+0=0+0$$

$$\boxed{z=0}$$

Thm 3: "negatives are unique" i.e. if $x+y=0$ & $x+z=0$ then $y=z$

So the notation $(-x)$ is usable and so subtraction makes sense.

Thm 4: $0_{\text{off}}(\text{vector})$

\uparrow vector

a. $0x_v = 0_v \rightarrow$ vector

b. $a \cdot 0_v = 0_v$

scalar

c. $(-a)x = a(-x) = -(ax)$

Thm 5: If x_i ($i=1, \dots, n$) are in V then

Follows from v5.1 & v5.2

" $\sum x_i$ " = $x_1 + x_2 + \dots + x_n$ make sense whichever way you parse it.
different

How many times you can add 3 elements? $(\cdot, \cdot), \cdot$ 2.3!

4 "

5. 4!

5 "

14. 5!

6 "

42. 6!

A soccer match ends with the results 3-3, and it is known that team B never led. How many histories could this match have?

① How many? ② Why did he ask? ③ Find a formula $3 \rightarrow n$

Def: let V be a v.s. a subspace of V is a subset W of V which is a vector space in itself under the operations it inherits for V .

Thus $W \subset V$ is a subspace of V iff (if and only if) \Leftrightarrow

$$\textcircled{1} \quad \forall x, y \in W \quad x+y \in W$$

and

$$\textcircled{2} \quad \forall a \in F \quad \forall x \in W \quad ax \in W$$

$$\textcircled{3} \quad 0 \in W \quad (W \text{ is not } \emptyset)$$

Part 1 \Rightarrow

Assume W is a subspace, if $x, y \in W$: then $x+y \in W$ because W is a v.s. in itself, likewise for aw .

\Leftarrow Assume $W \subset V$ for which $\begin{array}{c} x+y \in W \\ ax \in W \end{array}$

Need to show that W is a v.s. + & are clearly defined on W so we

just need check VS1-VS8

VS1 hold in V hence in W

VS2 " "

VS3 pick any $x \in W$ $0 = 0 \cdot x \in W$ by 2

VS4 given x in W , take $y = (-1)x \in W$ and $x+y=0$

VS5

{ same

VS8

Example:

Def: If $A \in M_{m \times n}(F)$ the "transpose" of A , A^t is the matrix

$$(A^t)_{ij} = A_{ji} \quad \text{i.e. } \begin{pmatrix} 2 & 3 & \pi \\ 7 & 8 & -2 \end{pmatrix}^t = \begin{pmatrix} 2 & 7 \\ 3 & 8 \\ 7 & -2 \end{pmatrix}$$

then 1. $A^t \in M_{n \times m}(F)$

2. $(A^t)^t = A$

3. $(A+B)^t = A^t + B^t$

4. $(\gamma A)^t = \gamma (A^t) \quad \forall \gamma$

Def: $A \in M_{n \times n}(F)$ is called "symmetric" if $A^t = A$

Claim:

$V = M_{n \times n}(F)$ a v.s.

$$W = \left\{ \begin{array}{l} \text{symmetric} \\ A \in V \end{array} \right\} = \left\{ A \in V : A^t = A \right\}$$

then W is a subspace of V

Prf:

① Need to show that if $A \in W \& B \in W$

then $A+B \in W \quad A^t = A, B^t = B$

$$(A+B)^t = A^t + B^t = A+B$$

② $7A \in W: (7A)^t = 7A^t = 7A \Rightarrow 7A \in W$ (if $A \in W$)

③ $O_m = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow O^t = O \text{ so } O \in W$

Example 2:

$$V = M_{m \times n}(\mathbb{F})$$

$$A = (A_{ij}) \quad \text{tr } A = \sum_{i=1}^n A_{ii}$$

"trace of of A "

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 3 \\ 8 & 2 & 9 \end{pmatrix}$$

$$\text{tr } A = 1+5+9 = 15.$$

Properties of tr .

1. $\text{tr } O_m = 0$

2. $\text{tr } (A+B) = \text{tr } (A) + \text{tr } (B)$

3. $\text{tr } (7A) = 7 \cdot \text{tr } A$

Set $W = \{A \in V : \text{tr } A = 0\} = \left\{ \begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix}, \dots \right\}$

Claim: W is a subspace

Indeed 1. $A, B \in W \Rightarrow \text{tr } A = 0 = \text{tr } B$

$$\text{tr } (A+B) = \text{tr } (A) + \text{tr } (B) = 0 + 0 = 0 \quad \text{so } A+B \in W$$

② $A \in W \quad \text{tr } A = 0$

$$\text{tr } 7A = 7 \cdot \text{tr } A = 7 \cdot 0 = 0 \quad \text{so } 7A \in W$$

⑤ $\text{tr } O_m = 0 \quad O_m \in W$

Example 3:

$$W_3 = \left\{ A \in M_{m \times n}(F) : \text{tr } A = 1 \right\}$$

Not a subspace!

$$\text{tr } (A+B) = \text{tr } A + \text{tr } B = 1+1=2 \quad A, B \in W_3 \quad \text{so } A+B \notin W_3$$

Thm:

The intersection of two subspaces of the same space is always a subspace.

Assume W_1, CV is a subspace of V ,

$W_2 \subset V$ is a subspace of V ,

then $W_1 \cap W_2 = \left\{ x : \begin{array}{l} x \in W_1 \\ \text{and} \\ x \in W_2 \end{array} \right\}$ is a subspace.

However, $W_1 \cup W_2 = \left\{ x : \begin{array}{l} x \in W_1 \\ \text{or} \\ x \in W_2 \end{array} \right\}$ is most often not a subspace.

Prf:

Assume $x, y \in W_1 \cup W_2$ that is,

$$x \in W_1, \quad x \in W_2 \quad y \in W_1, \quad y \in W_2$$

$x+y \in W_1$, as $x, y \in W_1$ & W_1 is a subspace

$x+y \in W_2$ as $x, y \in W_2$ & W_2 is a subspace

So $x+y \in W_1 \cap W_2$