models are isomorphic to the standard model given above. Suppose
$$\begin{tikzcd} M \arrow[dr, "i_1'"] & \\ & M \oplus' N \\ N \arrow[ur, "i_2'"'] & \end{tikzcd}$$
is another model of a direct sum.

Then,
$$\begin{tikzcd}[column sep=large] M \arrow[dr, "i_1'"] \arrow[drr, bend left, "i_1 : m \mapsto (m,0)"] & & \\ & M \oplus' N \arrow[r, dashed, shift left, "\exists! \gamma'"] & M \oplus N \arrow[l, dashed, shift left, "\exists! \gamma"] \\ N \arrow[ur, "i_2'"'] \arrow[urr, bend right, "i_2 : n \mapsto (0,n)"'] & & \end{tikzcd}$$

$i_1 = \gamma' \circ i_1'$ and $i_1' = \gamma \circ i_1$

$i_2 = \gamma' \circ i_2'$ $\quad$ $i_2' = \gamma \circ i_2$

It suffices to show that $\gamma' = \gamma^{-1}$

Now, $i_1 = \gamma' \circ i_1' = \gamma' \circ (\gamma \circ i_1) = (\gamma' \circ \gamma) \circ i_1$ So, $i_1(m) = (\gamma' \circ \gamma)(m,0)$ Since $\gamma' \circ \gamma$ is

$i_2 = \gamma' \circ i_2' = \gamma' \circ (\gamma \circ i_2) = (\gamma \circ \gamma') \circ i_2$ $\quad$ $i_2(n) = (\gamma' \circ \gamma)(0,n)$

an R-mod morphism, we have $(\gamma' \circ \gamma)(m,0) + (\gamma' \circ \gamma)(0,n) = (\gamma' \circ \gamma)(m,n)$. But $(m,n) = i_1(m) + i_2(n)$ $= (\gamma' \circ \gamma)(m,n)$. So, $\gamma' \circ \gamma = id_{M \oplus N}$. Similarly, $i_1' = (\gamma \circ \gamma') \circ i_1'$, $i_2' = (\gamma \circ \gamma') \circ i_2'$, so

$i_1'(m) = (\gamma \circ \gamma')(i_1'(m))$ $\quad$ So $i_1'(m) + i_2'(n) = (\gamma \circ \gamma')(i_1'(m)) + (\gamma \circ \gamma')(i_2'(n)) = (\gamma \circ \gamma')(i_1'(m) + i_2'(n))$.

$i_2'(n) = (\gamma \circ \gamma')(i_2'(n))$

Thus, to conclude that $\gamma \circ \gamma' = id_{M \oplus' N}$, we only NTS that

$M \oplus' N = \{ i_1'(m) + i_2'(n) : m \in M, n \in N \}$. "$\supseteq$" is clear. Fix any element $x \in M \oplus' N$.

$\gamma'(x) \in M \oplus N$, so $\gamma'(x) = (\underbrace{\gamma_1'(x)}_{\in M}, \underbrace{\gamma_2'(x)}_{\in N}) \in M \times N$. NTS: $x = i_1'(m) + i_2'(n)$ for some $m \in M$, $n \in N$.

$i_1(\gamma_1'(x)) + i_2(\gamma_2'(x)) = (\gamma_1'(x), \gamma_2'(x)) = \gamma'(x)$ $\quad \therefore$ $\gamma \circ \gamma'(x) = x = \gamma(i_1(\gamma_1'(x))) + \gamma(i_2(\gamma_2'(x)))$

i.e. $x = i_1'(\gamma_1'(x)) + i_2'(\gamma_2'(x))$, as needed. I.e. $M \oplus' N \subseteq \{ i_1'(m) + i_2'(n) : m \in M, n \in N \}$.

$\therefore M \oplus' N = \{ i_1'(m) + i_2'(n) : m \in M, n \in N \}$ and so we conclude that $\gamma \circ \gamma' = id_{M \oplus' N}$.

So, γ is a bijective R-mod morphism, and we conclude that $M \oplus N \cong M \oplus' N$. //

Proof of Thm 2. Take $M \oplus N$ to denote our standard model, and suppose another direct sum
$$\begin{tikzcd} M \arrow[dr, "i_1'"] & \\ & M \oplus' N \\ N \arrow[ur, "i_2'"'] & \end{tikzcd}$$
is given. Draw:
$$\begin{tikzcd}[column sep=large] M \arrow[dr, "i_1'"] \arrow[drr, bend left, "id"] & & \\ & M \oplus' N \arrow[r, dashed, "\exists! \gamma"] & M \\ N \arrow[ur, "i_2'"'] \arrow[urr, bend right, "0"'] & & \end{tikzcd}$$

So, $id_M = \gamma \circ i_1'$

$\therefore m = id_M = \gamma \circ i_1'(m)$

Suppose $i_1'(m_1) = i_1'(m_2)$. Then, $m_1 = \gamma \circ i_1'(m_1) = \gamma \circ i_1'(m_2) = m_2$. So i_1' is injective. By symmetry (aka draw a symmetric diagram for i_2') i_2' is injective. $\square$