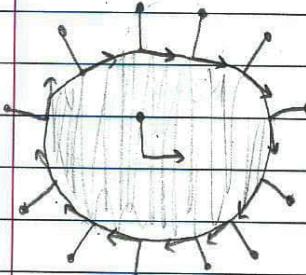


The fact that a manifold can't have 1 orientation is left out of the proof because I ran out of space but it's very obvious that the one orientation can be reversed to give a 2nd. A similar proof on Munkres' orientation is walked through in § 6.34 of Munkres.

Now that we know that M has only 2 Munkres orientations & 2 Dnor orientations, it's believable that Dnor & Munkres orientations have a natural correspondence. A Dnor & Munkres' orientation correspond if the pushforward of the standard basis of \mathbb{R}^k by the patches of the Munkres orientation agrees with the orientations chosen for $T(M)$ by the Dnor orientation. Essentially, these two notions are the same. We will refer to a Munkres orientation as an orientation from now on but keep in mind there is a corresponding Dnor orientation.

*An oriented manifold is a (M, \mathcal{O}) where M is a k-d manifold in \mathbb{R}^n & \mathcal{O} is an orientation of M Thru ∂M is orientable.

pf: The induced Dnor orientation of ∂M is such that prepending the vector pointing out of M of an $x \in \partial M$ will turn ∂_x in ∂M into ∂_x in M . We will focus on proving this orientation exists to build intuition.



Let $\{\alpha_i\}_{i \in I} = \mathcal{O}$ when I is some indexing set. $\forall x \in \partial M \exists i \in I / \alpha_i$ is a patch onto x . What we want is for $(\alpha_i^{-1}(x); -e_k) \neq \alpha_i^{-1}(x); e_1, \dots, \alpha_i^{-1}(x); e_{k-1}) \sim (\alpha_i^{-1}(x); e_j)_{j=1}^k$ as $(\alpha_i^{-1}(x); e_j)_{j=1}^k$ is in the Dnor orientation at x & $\alpha_i^{-1}(x); -e_k$ is the vector pointing out of M . We'll choose the identity of the \pm to achieve this. The Cob matrix from the 2nd to the 1st basis is clearly $\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$ which has determinant $(-1)^k$ if we choose + & $(-1)^{k+1}$ if we choose -.

So let \mathcal{O}_x be the class of $(-1)^k \alpha_i^{-1}(x); (\alpha_i^{-1}(x); e_1), \dots, (\alpha_i^{-1}(x); e_{k-1})$. Note this is just the pushforward of the std. basis of $\partial \mathbb{H}^k$ b, $\beta_i = \alpha_i \circ r(\cdot)$ if k is odd & $\beta_i = \alpha_i(\cdot)$ if k is even, when $r: \mathbb{H}^k \rightarrow \mathbb{R}^k$. If $(\beta_i^{-1}(\beta_i^{-1}(y); e_j))_{j=1}^{k-1} \in \mathcal{O}_y$, $\forall y \in M$ when β_i is defined, we can set $F_{ij} = (\beta_i^{-1}(\beta_i^{-1}(\cdot); e_j))_{j=1}^{k-1}$ & we have our vector fields & thus our Dnor orientation.

Copied
from Munkres
Thrm 34.1

Thus, if we show $\forall x, y \in M, (\beta_i^{-1}(\beta_i^{-1}(y); e_j))_{j=1}^{k-1} \sim (\beta_j^{-1}(\beta_i^{-1}(y); e_j))_{j=1}^{k-1}$, we're done. We'll do this by showing β_{ix}, β_{iy} overlap positively & the rest can be filled in from two proof ago.

If k is even, $D\beta_i^{-1} \circ \beta_j(z) = D(\alpha_i^{-1}(\cdot)) \circ \alpha_j^{-1}(\cdot)(z) = D\phi_{\alpha_i^{-1}, \alpha_j^{-1}}(\cdot)|_z$ where $\phi: \mathbb{R}^k \rightarrow \mathbb{R}^{k-1} / \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \end{pmatrix}$

So $D(\beta_i^{-1} \circ \beta_j)(z) = D\phi_{\alpha_i^{-1}, \alpha_j^{-1}}(\cdot) D(\alpha_j^{-1}(\cdot)) D(\alpha_i^{-1}(\cdot))|_z = [I_{k-1}] D(\alpha_i^{-1}(\cdot)) [I_{k-1}]$ for appropriate

Note if k is odd, there's cancel out & we get $D\beta_i^{-1} \circ \beta_j(z) = D\alpha_i^{-1} \circ \alpha_j^{-1}(\cdot)$ without the last row or column.

$\alpha_i^{-1} \circ \alpha_j$ is the transition function so $D(\alpha_i^{-1} \circ \alpha_j)(z) = (0 \dots 0x)|_{x>0}$ by expanding at x , we see

$\det D\beta_i^{-1} \circ \beta_j(z) = \lambda \det D\alpha_i^{-1} \circ \alpha_j^{-1}(\cdot) > 0$ as $\alpha_i^{-1} \circ \alpha_j$ overlap positively \blacksquare