Implicit Fn. THM
Inverse Fn. THM.
CHAIN RULE

MAT237 Midterm 1 Review

CHAIN RULE

THM. $f: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R}^m \to \mathbb{R}^p$, then: $D(g\circ f) = \underbrace{\underbrace{Dg(f(a))}_{p\times m} \cdot \underbrace{Df(a)}_{m\times n}}_{p\times n.}$

Ex: $\mathbb{R}_t \xrightarrow[t\mapsto (t,t)]{f} \mathbb{R}^2_{x,y} \xrightarrow[(x,y)\mapsto x^y]{g} \mathbb{R}$

$(g\circ f)(t) = g(f(t)) = g(t,t) = t^t.$

$D(g\circ f)(t) = ((t^t)').$

$Df(t) = \begin{pmatrix} \frac{\partial t}{\partial t} \\ \frac{\partial t}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad Dg\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} yx^{y-1} & x^y \log x \end{pmatrix}$

$Dg(f(t)) = Dg(t,t) = \begin{pmatrix} t t^{t-1} & t^t \log t \end{pmatrix} = B$

$Df(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = A.$

$D(g\circ f)(t) = BA = \begin{pmatrix} t t^{t-1} & t^t \log t \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boxed{t^t + t^t \log t}.$

Corollary 1. If f, g are C^r, then $g\circ f$ is C^r.

Corollary 2. If $f, g: \mathbb{R}^n \to \mathbb{R}^n$, f is differentiable at a, g is differentiable at $f(a) = b$, and $g(f(x)) = x$ near a, then $Dg(f(a)) = [Df(a)]^{-1}$.

Proof. Compute $D(g\circ f)(a)$:

$D(g\circ f)(a) = D(I)(a) = I_{n\times n}. \Rightarrow Dg(f(a)) = [Df(a)]^{-1}$!

INVERSE FUNCTION THM

f is one-to-one & $Df(x)$ is nonsingular for all $x \in A$

THM. If $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable (C^r) near $a \in \mathbb{R}^n$, and $Df(a)^{-1}$ exists, then there is a nbd U of a and V of $b = f(a)$ such that $f|_U : U \to V$ is invertible.
Furthermore, if f is C^r, then $(f|_U)^{-1}$ is C^r.

MEAN VALUE THM

THM. If $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable on the line ℓ between a & b, then there is a point $c \in \ell$, such that $\underbrace{f(b) - f(a)}_{\mathbb{R}} = \underbrace{Df(c)}_{1\times n} \cdot \underbrace{(b-a)}_{n\times 1}$

JELLY RIGIDITY

Technical Lemma. f is jelly-rigid near a: $f(y) - f(x) \sim (y-x)$, i.e.
$\forall \epsilon > 0$, $\exists$ a nbd $J = J_\epsilon = U(0;\delta)$, s.t. $\forall x, y \in J$: $\|f(y) - f(x)\| < \epsilon \|y - x\|$.

Proof of MVT in $\mathbb{R}^n$.

Consider $g: [0,1] \to \mathbb{R}$. Then $g(t) = f(a + t(b-a))$.

By 1-dim MVT: $g(1) - g(0) = g'(t_0)$, $\exists t_0 \in [0,1]$.

WTS $f(b) - f(a) = Df(c) \cdot (b-a)$

$\uparrow$

$a + t_0(b-a)$.

Proof of Technical Lemma.

Find $c_1, \ldots, c_n$ btw. x and y such that: D_i has all entries $\leq \epsilon/n$.

$$f_i(y) - f_i(x) = Df(c_i) \cdot (y-x) = (I + D_i)_i (y-x)$$

$$= y_i - x_i + d_{ii}(y-x)$$

small

$$\| f_i(y) - f_i(x) - (y_i - x_i) \| \leq n \, |d_{ij}| \, \|y-x\|$$

$< n \cdot \epsilon/n \cdot \|y-x\|$ on all of J, provided J small enough.

Then, $\|f(y) - f(x) - (y-x)\| \leq \epsilon \|y-x\|$. $\square$.

Proof of Inverse Function Theorem.

WLOG $Df(a) = I$, $a = b = 0$, and $J_\epsilon = U(a; \delta_\epsilon)$.

Part I. f is one-to-one on $J_{0.1}$.

Part II. f is onto on $0.4\,J_{0.1}$.

Part III. Take $V = 0.4\,J_{0.1}$ and $U = f^{-1}(V) \subseteq J_{0.1}$, $f|_U$ is 1-1, onto by I, II.

So, $(f|_U)^{-1}$ exists and is continuous.

Part IV. $(f|_U)^{-1}$ is differentiable at $a = 0$.

Part V. $(f|_U)^{-1}$ is differentiable near $a = 0$.

Part VI. f^{-1} is C^r near a.

IMPLICIT FUNCTION THM

THM. Given a C^r function $f: \underset{x_1,\dots,x_n}{\mathbb{R}^n} \times \underset{y_1,\dots,y_k}{\mathbb{R}^k} \to \underset{\text{unknowns}}{\mathbb{R}^k}$ and $(a,b) \in \mathbb{R}^n \times \mathbb{R}^k$ such that $f(a,b)=0$ [near (a,b)] and $\frac{df}{dy}$ is nonsingular, there is a unique C^r function $g: \{\text{nbd } U \text{ of } a\} \to \{\text{nbd of } b\}$ such that $g(a)=b$ and $\forall z \in U$, $f(z,g(z))=0$; furthermore $Dg = -\underbrace{\underbrace{\left(\frac{\partial f}{\partial y}\right)^{-1}}_{k\times k} \cdot \underbrace{\left(\frac{\partial f}{\partial x}\right)}_{k\times n}}_{k\times n}$.

evaluated at $(x, g(x))$.

Proof. Let $f(z,y)=0 \iff \begin{cases} x = z \\ f(x,y)=0 \end{cases}$ unknown x, y.

So now with $H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ f(x,y) \end{pmatrix}$ $\quad H: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k$

$\iff H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}$.

$H\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ f(a,b) \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ and H is C^r.

$$DH\begin{pmatrix} a \\ b \end{pmatrix}; \quad DH = \begin{pmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_1}{\partial y} \\ \frac{\partial H_2}{\partial x} & \frac{\partial H_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} I & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

must be invertible aka nonsingular.

$\Rightarrow \exists H^{-1}$ in some nbd of (a,b).

Set $g(z) = \pi_2\left(H^{-1}\begin{pmatrix} z \\ 0 \end{pmatrix}\right)$; w/ $\pi_2: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$

$(x,y) \mapsto x$

Easy to check $g(a)=b$ and $f(x,g(x))=0$ for all x.

$0 = f(x, g(x))$

$\downarrow$ take D

$0 = D[f(x,g(x))]$

$\Rightarrow 0 = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} I \\ Dg \end{pmatrix}$

$\to \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot Dg = 0 \Rightarrow Dg = -\overbrace{\underbrace{\left(\frac{\partial f}{\partial y}\right)^{-1}}_{k\times k} \cdot \underbrace{\left(\frac{\partial f}{\partial x}\right)}_{k\times n}}^{k\times n}$.

evaluated at $(x, g(x))$

$\square$