Def: A "subspace" W of a v.s. V is a subset of V which is also a v.s., using the same operation as in V, but restricted to W

eg: $V = \mathbb{R}^2 = \{\begin{pmatrix} a \\ b \end{pmatrix}\}$ $W = \{\begin{pmatrix} a \\ b \end{pmatrix} : a+b=0\}$

$\underset{W}{\underset{\Uparrow}{\begin{pmatrix} a \\ b \end{pmatrix}}}$ $\underset{W}{\underset{\Uparrow}{\begin{pmatrix} c \\ d \end{pmatrix}}}$ $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \underset{V}{\underset{\Uparrow}{\begin{pmatrix} a+c \\ b+d \end{pmatrix}}}$

$(a+b) + (c+d) = 0$ ✓.

reminder: $\underset{\text{subset}}{S} \subset \underset{\text{v.s.}}{V}$

$\text{span}(S) = \{u : u \text{ is a linear combination of vectors in } S\}$

$= \{u : \exists\, \alpha_i \in F,\ u_i \in S \text{ s.t. } u = \sum_{i=1}^{n} \alpha_i u_i\}$

always a subspace.

S "generates" or "span" V if $V = \text{span}(S)$

Example In $V = M_{2\times2}(\mathbb{F}_3)$.

$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $M_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $N_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $N_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $N_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Claims

$M_1 --- M_4$ generate (span) V, indead.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Aside

1. If W is a subspace of V and $S \subset W$, then $\operatorname{span}(S) \subset W$

PF: W is closed under addition & multiplication by scalars, hence under l.s.

2. If $S_2 \subset \operatorname{span}(S_1)$ then $\operatorname{span}(S_2) \subset \operatorname{span}(S_1)$

Sub example: $u_1, u_2 \in S_1$

$$S_2 = \left\{ \begin{matrix} 7u_1 + 3u_2 = v_1 \\ 2u_1 - u_2 = v_2 \end{matrix} \right\}$$