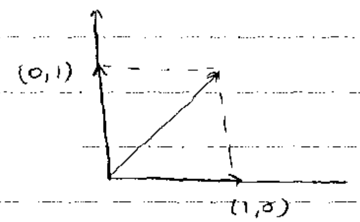
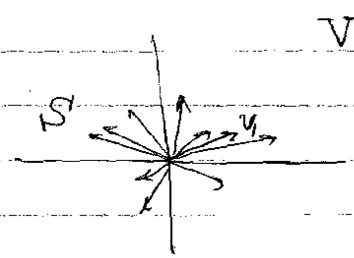
Tutorial

Basis = maximal lin. ind. subset

Basic ex $\mathbb{R}^2$: plane (subspace is the line that passes through origin)





① Let V be a v.s. $\dim V = \alpha < \infty$

Let $S \subset V$ be a subset of which generates V, i.e. $\text{span}\{S\} = V$ (S can be infinite)

(a) Show that $\exists$ subset in S which is a basis of V.

Prf: Take any $v_1 \in S \subset V$ if all the other vectors in S are lin. dep.

on v_1, then $S = \text{span}\{v_1\} \Rightarrow V = \text{span}(S) = \text{span}(\text{span}\{v_1\}) = \text{span}\{v_1\}$

Then $V = \text{span}\{v_1\} \Leftrightarrow \dim V = 1$ $(n = 1)$ $\Rightarrow$ v_1 is a basis of V.

If not, then $\exists\, v_2 \in S$, s.t. v_1 & v_2 are lin. indep.

Consider $\text{span}\{v_1, v_2\} \subset V$. If any $v_3 \in S$ is lin. dep. on v_1, v_2.

Then $\text{span}\{v_1, v_2\} = \text{span}(S) = V \Rightarrow \{v_1, v_2\}$ is a basis of $V \Leftrightarrow \dim V = 2$ $(n=2)$

Continuing the process we get a finite set $\{v_1, \dots, v_n\}$ which are
because $\dim V < \infty$

lin. ind. & $\text{span}\{v_1, \dots, v_n\} = V$ & maximal

(b) Show that S contains at least n vectors

prove: trivial $v_1, ---, v_n \in S$

assume (a) is not known,

assume that S contains of m vectors, $m<n$ $S = \{w_1, --- w_m\}$

Then $\text{span}\{S\} = \text{span}\{w_1, ---, w_m\} = V$ $\lightning$

because $m < n = \dim V$.

2) Let W_1, W_2 be v.s

$$Z := \{(w_1, w_2) \mid w_1 \in W_1, w_2 \in W_2\}$$ be a v.s. with usual operations.

Let $\dim W_1 = n_1$ $\dim W_2 = n_2$ $\Rightarrow$ $\dim Z = ?$

Consider the bases of W_1, W_2

$\{e_1, --- e_{n_1}\}$ - basis of W_1

$\{f_1, --- f_{n_2}\}$ - - " ——

& consider the set of vectors in Z:

$$\{(e_1, 0), ---, (e_{n_1}, 0), (0, f_1), --- (0, f_{n_2})\} := S \subset Z$$

$*$ el's $= n_1 + n_2$

Then S is a basis of Z. Indeed, take any $(w_1, w_2) \in Z$

Since $w_1 \in W_1$, there are constants $a_1, \text{——}, a_{n_1}$ s.t.

$$w_1 = a_1 e_1 + \text{— —} + a_n e_n$$

Since $w_2 \in W_2$, there are constanst $b_1, \text{——} b_{n_2}$ s.t

$$w_2 = b_1 f_1 + \text{- - -} + b_{n_2} f_{n_2}$$

Then

$$(w_1, w_2) = a_1(e_1, 0) + \cdots - + a_n(e_{n_1}, 0) + b_1(0, f_1) + \text{——} + b_{n_2}(0, f_{n_2})$$

Conclusion span$(S) = Z$

It remains to show that all the el's S are li ind.

Assume $\exists\, \alpha_1, \text{——}, \alpha_{n_1}, \beta_1, \text{——} \beta_{n_2}$ s.t.

$$\alpha_1(e_1, 0) + \text{- - -} + \alpha_{n_1}(e_{n_1}, 0) + \beta_1(0, f_1) + \text{- —} + \beta_{n_2}(0, f_{n_2}) = 0$$

$$\parallel$$

$$(\alpha_1 e_1 + \text{- —} + \alpha_{n_1} e_{n_1}, \ \beta_1 f_1 + \text{——} + \beta_{n_2} f_{n_2})$$

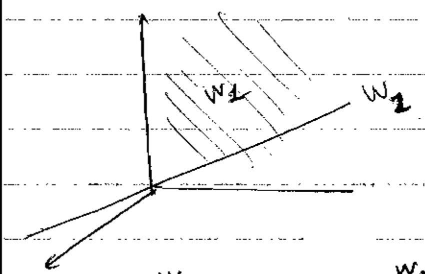
$$\alpha_1 = \text{- - -} = \alpha_{n_1} = 0 \qquad 0 \Rightarrow \beta_1 = \text{- - -} \beta_{n_2} = 0$$

(3) In the next problem set we need to prove

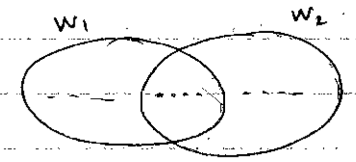
$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \quad (*)$$

Explation: W_1, W_2 are linear subspace's of V

$$W_1 + W_2 = \{x + y \mid x \in W_1, y \in W_2\}$$



$\dim(W_2+W_1) = \dim W_1 + \dim W_2 - 1$ (since they intersect)



Prove that

(a) $\dim(W_1 \cap W_2) \leq n$ $\quad \dim W_1 = m$, $\dim W_2 = n$ $\quad n \leq m$

proof: $W_1 \cap W_2$ is a vector space. Let $\{e_1, \_\_\_, e_k\}$, basis of $W_1 \cap W_2$

(i.e. $\dim(W_1 \cap W_2) = k$

Now $W_1 \cap W_2 \subset W_1 \Rightarrow \dim(W_1 \cap W_2) \leq \dim W_1 = m$.

& $W_1 \cap W_2 \subset W_2 \Rightarrow \dim(W_1 \cap W_2) \leq \dim W_2 = n$.

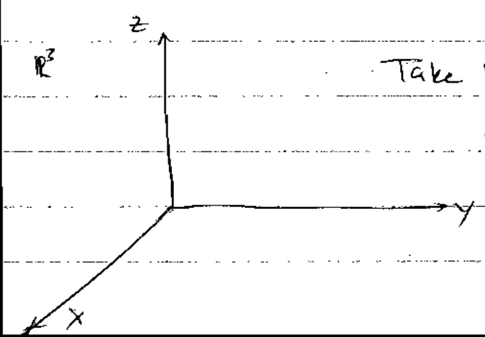
$\Rightarrow k \leq m$ & $k \leq n$

(b) $\dim(W_1 + W_2) \leq n+m$

this by (*) [look @ 3]

4) find an example of $W_1, W_2 \subset \mathbb{R}^3$ s.t.

(a) $m > n > 0$, $\dim(W_1 \cap W_2) = n$



Take W_1 = z-y plane

W_2 = any line through origin in W_2

(b) $\dim(W_1 + W_2) = m+n$

Take either two lines or a plane and a line not in this plane
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad W_1 \qquad\qquad\qquad W_2$

(c) $m \geq n$ & $\dim(W_1 \cap W_2) < n$, $\dim(W_1 + W_2) < m+n$ (in $\mathbb{R}^3$)

Take $W_1 = x\text{-}y$ plane, $W_2 = z\text{-}y$ plane, $W_1 \cap W_2 = y$-coord.
$\dim 1 < n = 2$

$\dim(W_1 + W_2) = n+m-1 < n+m$

5) Let $W_1 \subset P(\mathbb{R})$, polynomials over $\mathbb{R}$ be a subspace consisting of

$W_1 = \{ a_0 + a_1 x + \cdots + a_n x^n \mid a_i = 0$ whenever i is odd $\}$
$n \in \mathbb{N} \qquad i = 2k+1$

& $W_2 \subset P(\mathbb{R})$ is the subspace

$W_2 = \{ b_0 + b_1 x + \cdots + b_m x^m \mid b_i = 0$ whenever i is even, $i = 2k \}$
$m \in \mathbb{N}$

What is $\dim(W_1 \cap P_n(\mathbb{R})) = ?$

$\dim(W_2 \cap P_n(\mathbb{R})) = ?$

Sol'n: What is the natural basis of W_1?

It is $S_1 = \{1, x^2, \ldots, x^{2k}, \ldots\}$

Likewise the basis of W_2 is

$S_2 = \{x, \ldots, x^{2k+1}, \ldots\}$

looking for $\mathbb{R}/\mathbb{Q}$ basis

$\mathbb{R} \to$ scalars

Then: the basis of $W_1 \cap P_n(\mathbb{R})$:

is $\{1, x^2, \ldots, x^{2k_1}\}$, where $2k_1 \le n$ & $2(k+1) > n$

So if n = even $\Rightarrow \dim(W_1 \cap P_n(\mathbb{R})) = \frac{n}{2} + 1$

if n = odd $\Rightarrow \dim(W_1 \cap P_n(\mathbb{R})) = \frac{n+1}{2} + 1$

* the basis of $W_2 \cap P_n(\mathbb{R})$: $\{x, x^3, \ldots, x^{2k_2+1}\}$, $2k_2 + 1 \le n$

$2(k_2+1)+1 > n$

So if n = even $\Rightarrow \dim(W_2 \cap P_n(\mathbb{R})) = \frac{n}{2}$

if n = odd $\Rightarrow \dim(W_2 \cap P_n(\mathbb{R})) = \frac{n-1}{2}$

Linear Transformations

Lee 17.10.06.

... but first

1. (fishy) Theorem: Every vector space has a basis.

Examples R/Q a. $\{1, \sqrt{2}, \pi, e, \ldots$ } we don't know if $\pi + e$ are rational !

$\pi + e = r.1$

$e = r.1 - \pi$

2. Lagrange Approximation

Let $(x_i)_{i=1}^{n+1} = (x_1, \ldots, x_{n+1})$ be distinct numbers in $\mathbb{R}$

Let $(y_i)_{i=1}^{n+1}$ be arbitrary numbers in $\mathbb{R}$.