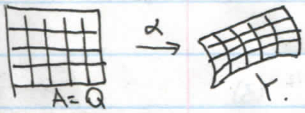
Jan. 16.

$A \subset \mathbb{R}^k$

Def. "A parametrized k-manifold in $\mathbb{R}^n$" is a C^1 map $\alpha: A \to \mathbb{R}^n$ where $A \subset \mathbb{R}^k$ is some open set. $\alpha(A) = Y$ "the manifold". α: the parametrized.

$\subset \mathbb{R}^n$



$A = Q$

Pick a fine partition P of Q.

$$\mathrm{Vol}(Y) \sim \sum_{R \in P} \mathrm{Vol}(\alpha(R)) \sim \sum_{R \in P} V(D\alpha(c)(d_1 - c_1)e_1, D\alpha(c)(d_2-c_2)e_2, \dots, D\alpha(c)(d_k - c_k)e_k).$$

$R = \Pi[c_i, d_i]$, $c = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$ $(*)$

Claim: $V(x_1, \dots, \lambda x_i, \dots x_k) = |\lambda| \cdot V(x_1, \dots, x_k)$.

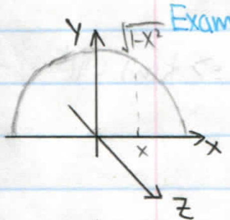
$$(*) = \sum_{R \in P} \prod_{i=1}^{k} |d_i - c_i| \cdot V(D\alpha(c)e_1, \dots D\alpha(c)e_k) \rightsquigarrow X = D\alpha(c)$$

$$= \sum_{R \in P} \mathrm{Vol}(R) \cdot V(D\alpha(c)) \sim \int_Q V(D\alpha)$$

$V(x_1, \dots, x_k) = |\det X^T X|^{1/2} = V(X)$ if $X \in M_{n \times k}$

Def. If Y is ... as above, $V(Y) = V(Y, \alpha) := \int_A V(D\alpha) = \int_A |\det((D\alpha)^T D\alpha)|^{1/2}$

"$V(Y, \beta)$?"



Example. Compute the amount of crust on a still ok. a spherical loaf of bread.

$A = (a,b)x \times (0, 2\pi)\theta$. $\quad \alpha(x, \theta) = \begin{pmatrix} x \\ \sqrt{1-x^2}\cos\theta \\ \sqrt{1-x^2}\sin\theta \end{pmatrix}$ $\quad D\alpha = \begin{pmatrix} 1 & 0 \\ \frac{-x\cos\theta}{\sqrt{1-x^2}} & -\sqrt{1-x^2}\sin\theta \\ \frac{-x\sin\theta}{\sqrt{1-x^2}} & \sqrt{1-x^2}\cos\theta \end{pmatrix}$

$(D\alpha)^T D\alpha = \begin{pmatrix} \frac{1}{1-x^2} & 0 \\ 0 & 1-x^2 \end{pmatrix}$ $\quad V(D\alpha) = 1$.

$\mathrm{Vol} = \int_A 1 = \mathrm{Area}(A) = 2\pi(b-a)$. $\quad [a,b] = [-1, 1] \rightsquigarrow$ surface area of a sphere is 4π.

$$\begin{tikzcd} A \subset \mathbb{R}^k \arrow[r, "\alpha"] & Y \subset \mathbb{R}^n \arrow[r, "f", "\text{cont.}"'] & \mathbb{R} \\ B \arrow[u, "g"] \arrow[ur, "\beta = \alpha \circ g"'] & & \end{tikzcd}$$

Thm. If $\alpha: A \to \mathbb{R}^n$ is a mfld & if $g: B \to A$ ($B \subset \mathbb{R}^k$, $A \subset \mathbb{R}^k$) is a diffisomorphism, then set $\beta = \alpha \circ g$ and then $\alpha(A) = \beta(B)$, and $V(Y, \alpha) = V(Y, \beta)$.

pf. $V(Y, \beta) = \int_B V(D_\beta) = \int_B V(D(\alpha \circ g)) = \int_B V((D\alpha) \circ g \cdot Dg) = \int_B |\det((Dg)^T \circ ((D\alpha)\circ g)^T (D\alpha) \circ g \cdot Dg)|^{1/2}$

$= \int_B |\det Dg| (V(D\alpha) \circ g) = \int_A V(D\alpha) = V(Y, \alpha)$.

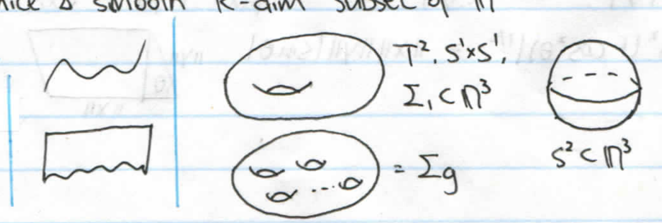
Jan 18

Def. $\int_Y f\,dV = \int_A (f \circ \alpha) V(D\alpha) \overset{\text{thm}}{=} \int_B (f \circ \beta) V(D\beta)$

M and $\partial M \leftarrow$ boundary of M.

M: "A nice & smooth k-dim subset of $\mathbb{R}^n$"

Examples:



$T^2, S^1 \times S^1$, $\Sigma_1 \subset \mathbb{R}^3$

$= \Sigma_g$

$S^2 \subset \mathbb{R}^3$

K-bottle $\subset \mathbb{R}^4$

$O(n) = \{A \in M_{n \times n} : A^T A = I\} \subset \mathbb{R}^{n^2}$

$O(2) = S^1 \cup S^1 \subset \mathbb{R}^4$

Def. A k-dim manifold (w/o boundary, of class C^r, $r \geq 1$) in $\mathbb{R}^n$, is a subset $M \subset \mathbb{R}^n$ s.t. each $p \in M$ has an open nbd V s.t. there is an open $U \subset \mathbb{R}^k$ & a C^r homomorphism $\alpha: U \to V$ whose differential $D\alpha(x)$ has rank for every $x \in U$.

Coordinate match