December 3rd, 2009

MAT240: Abstract Linear Algebra Lecture:

Problem:

Let V and W be finite dimensional vector spaces, let $\beta = (u_1 \dots u_n)$ and $\beta` = (u`_1 \dots u`_n)$ be bases of V and let $\gamma = (w_1 \dots w_n)$ and $\gamma` = (w`_1 \dots w`_m)$ be bases of W and let $T: V \to W$ be a linear transformation and let $A = [T]_\beta^\gamma$ & $A` = [A`]_{\beta`}^{\gamma`}$. Can we write A\` using A, without again looking at T?

Author's note: The diagram that follows is far too complex to transfer to a coherent PDF format. Please see notes posted by other students for the diagram. If no notes exist, this diagram is somewhat similar in spirit to the diagram found of page 105, Figure 2.2 of our text book.

Solution: Let $P = [I_w]_{\gamma`}^{\gamma} = \left([w`_1]_\gamma \mid \dots [w`_m]_\gamma\right)$, let $Q = [I_v]_{\beta`}^{\beta} = \left([u`_1]_\beta \mid \dots [u`_n]_\beta\right)$

then $A` = P^{-1}AQ$

eg. $T: R^2 \to R^2$ is given by $\begin{pmatrix} -7 & 3 \\ -18 & 8 \end{pmatrix} = A$ (relative to standard bases of both ie. $T = T_A$, $T_V = Av$). What's $[T]_\theta^\theta = A`$ where $\theta = \left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}\right)$?

Solution: $P = Q = [T_{R^2}]_\theta^{(e1,e2)} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$A` = P^{-1}AQ = P^{-1}AP$$

$$P^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\to P^{-1}A = P^{-1}A \to \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} -7 & 3 \\ -18 & 8 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

$$\to P^{-1}AP = A` \to \begin{pmatrix} -4 & 6 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

Another Method:

$$A` = [T]_{\theta}^{\theta} = ([T_{\theta 1}]_{\theta} | [T_{\theta 2}]_{\theta}) = \left[\begin{pmatrix} 2 \\ 6 \end{pmatrix}\right]_{\theta} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T_{\theta 1} = \begin{pmatrix} -7 & 3 \\ -18 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$T_{\theta 2} = \begin{pmatrix} -7 & 3 \\ -18 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + (-1) \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$