P56

17. Solution:

A is a skew-symmetric $n \times n$ matrix, then $\forall i, j$,

a_{ij} in A $a_{ij} = -a_{ji}$ $(A^T = -A)$

Then we can form a subset of W, which contains all matrices $A_{k,l}$, for all $1 \le i \le j \le n$, $1 \le k < l \le n$, $a_{ij} = \begin{cases} 1 & i = k \quad j = l \\ 0 & i \ne k \quad j \ne l \end{cases}$

$a_{ji} = \begin{cases} -1 & i = k, \ j = l \\ 0 & i \ne k, \ j \ne l \end{cases}$ $\quad i, j, k, l \in \mathbb{N}$

We are now trying to prove $\{A_{k,l}\}$ is a basis of W

$\{A_{k,l}\} = \left\{ A_{1,2} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ -1 & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix}, A_{1,3} = \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & & & \vdots \\ -1 & & \ddots & & \\ \vdots & & & & \\ 0 & \cdots & & & 0 \end{pmatrix}, \cdots, A_{n-1,n} = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ & & 0 & 1 \\ 0 & \cdots & -1 & 0 \end{pmatrix} \right\}$

Then $\forall B$ in W, $B = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & & \vdots \\ \vdots & & \ddots & \\ -a_{1n} & \cdots & & 0 \end{pmatrix}$, we need to show B can be uniquely expressed by $\{A_{k,l}\}$, suppose that.

$B = a_{1,2} \cdot A_{1,2} + a_{1,3} \cdot A_{1,3} + \cdots + a_{n-1,n} \cdot A_{n-1,n}$

$= a_{1,2} \begin{pmatrix} 0 & 1 & \cdots & 0 \\ -1 & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix} + \cdots + a_{n-1,n} \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ & & 0 & 1 \\ 0 & \cdots & -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & a_{1,2} & \cdots & 0 \\ -a_{1,2} & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ & & 0 & a_{n-1,n} \\ 0 & \cdots & -a_{n-1,n} & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & a_{1,2} & \cdots & \cdots & 0 \\ -a_{1,2} & 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & a_{n-1,n} \\ 0 & \cdots & \cdots & -a_{n-1,n} & 0 \end{pmatrix}$$

Then we can get a system of linear equations

$$\begin{cases} a_{1,2} = a_{12} \\ -a_{1,2} = -a_{1,2} \\ \quad\vdots \\ a_{n-1,n} = a_{n-1n} \\ -a_{n-1,n} = -a_{n-1n} \end{cases} \Rightarrow a_{i,j} = a_{ij} \ (i<j)$$

a_{ij} is unique $\Rightarrow a_{i,j}$ is unique.

Hence $B = a_{12}A_{1,2} + \cdots + a_{n-1n}A_{n-1,n}$

i.e. each $M \in W$ can be uniquely expressed as a linear combination of vectors of $\{A_{k,l}\}$.

By Thm 1.8 on the textbook on P43

$\{A_{k,l}\}$ is a basis for W.

$\{A_{k,l}\}$ has $1+2+\cdots+(n-1) = \frac{n(n-1)}{2}$ elements

So W's dimension is $\frac{n(n-1)}{2}$

P57

26. Solution:

Assume subspace $W = \{f \in P_n(\mathbb{R}) : f(a) = 0\} \subseteq P_n(\mathbb{R})$

Then ? for all $f \in W$

$f(x) = (x-a)(a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$ $\forall a_i \in \mathbb{R}$

? Then we know all $g(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ $\forall a_i \in \mathbb{R}$ form $P_{n-1}(\mathbb{R})$

$P_{n-1}(\mathbb{R})$ has a standard basis $\{1, x, x^2, \cdots, x^{n-1}\}$,

which means $\beta = \{(x-a)\cdot 1, (x-a)\cdot x, (x-a)x^2, \cdots (x-a)\cdot x^{n-1}\}$ is a basis of W. ??

we are trying to show that in the following discussion

$f(x) = (x-a)(a_{n-1} \cdot x^{n-1} + \cdots + a_1 x + a_0)$

$= (x-a) \cdot a_{n-1} \cdot x^{n-1} + \cdots + (x-a)a_1 x + (x-a)a_0$

$= (x-a) \cdot x^{n-1} \cdot a_{n-1} + \cdots + (x-a)x \cdot a_1 + (x-a) \cdot 1 \cdot a_0$

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Then β spans W.

for $0 = a_{n-1}'(x-a) \cdot x^{n-1} + \cdots + a_0'(x-a) \cdot 1$

$\Rightarrow a_0' = \cdots = a_{n-1}' = 0$ i.e. β is linear independent.

$\Rightarrow \beta$ is a basis of W β has n elements.

Hence the dimension is n.