DISCLAIMER

Consult these solutions at your own risk! This assignment got 100 percent and the TA made no remarks.

MAT257: Problem Set 15

Section 29

1.

Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be C^r. By definition, the velocity vector to γ at t is the vector $(\gamma(t); D\gamma(t))$. Note that

$$D\gamma = \begin{bmatrix} \frac{\delta\gamma_1}{x} \\ \cdots \\ \frac{\delta\gamma_n}{x} \end{bmatrix} \in M_{n\times 1}(\mathbb{R})$$

Therefore $\gamma_*(t; e_1) = (\gamma(t); D\gamma(t) \cdot e_1) = (\gamma(t); [\frac{\delta\gamma}{\delta x_1}](t)) = (\gamma(t); D\gamma(t))$

Thus the velocity vector to γ at t is the vector $\gamma_*(t; e_1)$.

2.

Let $\alpha : A \to \mathbb{R}^n$ be C^r and $\gamma(t) = \alpha(x + tv)$. Let $\beta(t) = x + tv$. Note that $\alpha \circ \beta(t)$ is a C^r function and that $\gamma(t) = \alpha \circ \beta(t)$. Thus, by the Chain Rule, $D\gamma(t) = D\alpha(\beta(t)) \cdot D\beta(t) = D\alpha(x + tv) \cdot D\beta(t)$

By definition, the velocity vector to γ at t is the vector $(\gamma(t); D\gamma(t))$.

If $t = 0$, we have that

$(\gamma(t); D\gamma(t)) = (\alpha(x + tv); D\alpha(x + tv) \cdot D\beta(t))$

$= (\alpha(x); D\alpha(x) \cdot v) = \alpha_*(x; v)$

Therefore $\alpha_*(x; v)$ is the velocity vector to γ at $t = 0$.

4.

a) Suppose we are given a point p in $M - \delta M$, a k-manifold without boundary, and a corresponding tangent vector $(p; v)$. By definition, there is a co-ordinate patch $\alpha : U \to V$ such that $U \subset \mathbb{R}^k$ is open and $p \in V$. We also know, by definition, that $(p; v) = \alpha_*(x; w)$ for some tangent vector $(x; w)$ to $\mathbb{R}^k$.

Without loss of generality, assume that $x = 0 \in U$. This is a valid assumption, as the requisite translation T will be a "nice" diffeomorphism, so we can replace α in our argument by $\alpha \circ T$ with no change. Since U is open, it contains some metric ball A of radius ϵ about the origin. Define a function $\gamma(t) : (-\epsilon, \epsilon) \to \mathbb{R}^n$ by $\gamma(t) = \alpha(x + tw)|_{(-\epsilon,\epsilon)}$. Then $\alpha_*(x; w) = (p; v)$ is the velocity vector of $\gamma(t)$ at $t = 0$, as shown in (2). Additionally, γ is a parametrized curve as it is C^r, since it is the restriction of a C^r map, and its domain is open in $\mathbb{R}$.

Thus some γ exists, as required.

b)

Suppose there is a parametrized curve $\gamma(t) : (-\epsilon, \epsilon) \to \mathbb{R}^n$ such that $\gamma((-\epsilon, \epsilon)) \subset M$ and $(p; v)$ is its velocity vector at the point $t = 0$ where $\gamma(0) = p \in M - \delta M$.

Let $\alpha : U \to V$ be the co-ordinate patch of M containing $p = \gamma(0)$. Note that $U \subset \mathbb{R}^k$. Thus $(\alpha_*)^{-1} = (\alpha^{-1})_*$ takes the tangent vector $(p; v)$ of γ (and $\mathbb{R}^n$) to the tangent vector $(\alpha^{-1}(p); D\alpha^{-1}(p) \cdot v)$ of $\mathbb{R}^k$. This is a perfectly valid vector since α^{-1} is C^r, as α is a co-ordinate patch.

By definition,

$\alpha_*(\alpha^{-1}(p); D\alpha^{-1}(p) \cdot v) = ((\alpha \circ \alpha^{-1})(p); D\alpha(\alpha^{-1}(p)) \cdot [D\alpha^{-1}(p) \cdot v])$

By the Chain Rule, we have

$\alpha_*(\alpha^{-1}(p); D\alpha^{-1}(p)) = (p; D(\alpha \circ \alpha^{-1})(p) \cdot v) = (p; 1 \cdot v) = (p; v)$

However, being a co-ordinate patch of M, the image of α_* is exactly the tangent space of M. Thus $(p; v)$ must be a tangent vector to M, as well.