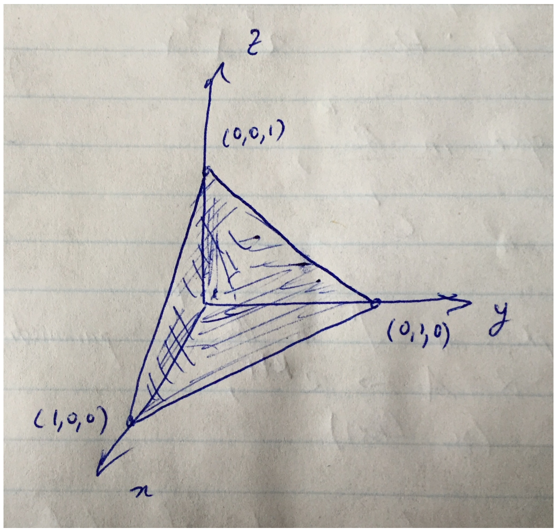
PSet 11: Partial Solutions

DISCLAIMER: I cannot claim that what I have written here constitutes a perfect solution. Certainly some mistakes are present; hopefully these mistakes aren't too severe. I hope that my answers may serve as a guide to you when studying for the final exam.

Problem 17.6. Let $U = \{(x, y, z) : x, y, z \geq 0 \text{ and } x + y + z \leq 1\}$, and let $g : U \to \mathbb{R}^3$ be the linear transformation defined by $g(x, y, z) = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x - z, 2x + y + z, 3x + 2y + z)$. The set $U \subset \mathbb{R}^3$ may be visualized as follows:



Now, $(f \circ g)(x, y, z) = x - z + 4x + 2y + 2z - 3x - 2y - z = 2x$. We note that g is $\mathcal{C}^1$, as its components are $\mathcal{C}^1$. We also observe that g maps this set U onto the tetrahedron S (in particular, it maps the unit vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ to the vectors $(1, 2, 3), (0, 1, 2), (-1, 1, 1)$, respectively). Moreover, since Dg is non-singular on U, then by the Inverse Function Theorem we have that g is invertible, and that the inverse function g^{-1} is $\mathcal{C}^1$. Now, by the Change of Variables Theorem, $\int_S f$ exists if and only if $\int_U (f \circ g)|\det Dg|$ exists, and that these integrals are equal in this case.

$$\begin{aligned} \int_U (f \circ g)|\det Dg| &= \int_U 2x && \text{as computed above} \\ &= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} 2x|\det Dg| dx dy dz \end{aligned}$$

The second line follows from Fubini's Theorem, and from the fact that the sloping surface is given by $x + y + z = 1$ so that when y, z are fixed, x can range from 0 to $1 - y - z$, and when z is fixed, y can range from 0 to $1 - z$. Now,

$$|\det Dg| = \left| \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} - \det \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \right| = 2$$

Hence,

$$\begin{aligned}
\int_U (f\circ g)|\det Dg| &= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} 4x dx dy dz \\
&= \int_0^1 \int_0^{1-z} 4(1-y-z)^2 dy dz \\
&= \int_0^1 \frac{2}{3}(y+z-1)^3|_0^{1-z} dz \\
&= \int_0^1 \frac{-2}{3}(z-1)^3 dz \\
&= \frac{-1}{6}(z-1)^4|_0^1 \\
&= \frac{1}{6}
\end{aligned}$$

□

Problem 20.2. We say that a linear transformation $h : \mathbb{R}^n \to \mathbb{R}^n$ preserves volume if for every rectifiable set $S \subset \mathbb{R}^n$, we have that $h(S)$ is rectifiable and that $v(h(S)) = v(S)$. Now, consider the linear transformation $g : \mathbb{R}^n \to \mathbb{R}^n$ defined by $g(x) = \begin{pmatrix} 2 & 0 & \\ 0 & 1/2 & O_{2,(n-2)} \\ O_{(n-2),2} & & I_{n-2} \end{pmatrix} x$, where $O_{p,q}$ denotes the zero matrix of size $p \times q$. Then,

$$\begin{aligned}
\det Dg &= \det \begin{pmatrix} 2 & 0 & \\ 0 & 1/2 & O_{2,(n-2)} \\ O_{(n-2),2} & & I_{n-2} \end{pmatrix} \\
&= 2 \cdot \frac{1}{2} \cdot 1^{n-2} \\
&= 1.
\end{aligned}$$

Hence, by Theorem 20.1, we have that g preserves n-dimensional volumes. Moreover, g is not an orthogonal transformation, by definition, as its columns do not form an orthonormal basis in $\mathbb{R}^n$. Hence, by Theorem 20.5b, g is not an isometry, as required.

□

Problem 21.3. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be the function $h(x) = \lambda x$, and let $\mathcal{P} = \mathcal{P}(x_1, \ldots, x_k)$ be a k-dimensional parallelopiped in $\mathbb{R}^n$. We claim that $v(h(\mathcal{P})) = |\lambda|^k v(\mathcal{P})$.

Proof. Let $X = \begin{pmatrix} x_1 & x_2 & \ldots & x_k \end{pmatrix}$. Now, by definition of the volume of a k-dimensional parallelopiped in $\mathbb{R}^n$, we have that $v(h(\mathcal{P})) = \left|\det((\lambda X)^T(\lambda X))\right|^{1/2} = \left|\det(\lambda^2 X^T X)\right|^{1/2}$. We now prove by induction that if M is a $m \times m$ matrix, then $\det(cM) = c^m \det M$. For the base case, when $m = 1$, $\det(cM) = cM = c^1 \det M$, as M has only one entry. Now, suppose that the result holds for $(m-1) \times (m-1)$ matrices, and consider a matrix $M = (a_{ij})$ of size $m \times m$. Then by the determinant formula,

$$\begin{aligned}
\det cM &= \sum_{i=1}^{m} (-1)^{i+1}(ca_{i1})\det(cM_{i1}) && \text{where each minor } cM_{i1} \text{ is of size } (m-1)\times(m-1) \\
&= \sum_{i=1}^{m} (-1)^{i+1}(ca_{i1})c^{m-1}\det M_{i1} && \text{by the inductive hypothesis} \\
&= c^m \sum_{i=1}^{m} (-1)^{i+1} a_{i1}\det M_{i1} \\
&= c^m \det M,
\end{aligned}$$

as required. Hence, since $X^T X$ defined above is a $k \times k$ matrix, we have,

$$\begin{aligned}
v(h(\mathcal{P})) &= \left|\det(\lambda^2 X^T X)\right|^{1/2} \\
&= \left|\lambda^{2k}\det(X^T X)\right|^{1/2} \\
&= |\lambda|^k v(\mathcal{P}).
\end{aligned}$$

□

Problem 22.2. We claim that the desired integral is given by $\int_{x\in A}\sqrt{1+\sum_{i=1}^{k}(\partial_i f(x))^2}$.

Proof. Since α is $\mathcal{C}^1$, then by the first definition in section 22 of Munkres we have that the volume of Y_α is defined as $v(Y_\alpha) = \int_A V(D\alpha)$. Now, $\alpha(x) = (x, f(x)) = (i(x), f(x))$ for all $x \in A$, where i is the identity function. Hence, $D\alpha = \begin{pmatrix} Di \\ Df \end{pmatrix} = \begin{pmatrix} I_k \\ Df \end{pmatrix}$. So,

$$\begin{aligned}
V(D\alpha) &= \left|\det(D\alpha)^T(D\alpha))\right|^{1/2} \\
&= \left|\det\left(\begin{pmatrix} I_k^T & Df^T \end{pmatrix} \cdot \begin{pmatrix} I_k \\ Df \end{pmatrix}\right)\right|^{1/2} \\
&= \left|\det\begin{pmatrix} I_k & Df^T Df \end{pmatrix}\right|^{1/2} \\
&= |1 + DfDf^T|^{1/2} && \text{by Dror's hint, that } \det(I_n + vw^T) = 1 + v^T w \quad \forall v, w \in \mathbb{R}^n \\
&= \sqrt{1 + \sum_{i=1}^{k}(\partial_i f)^2}
\end{aligned}$$

Hence,

$$\begin{aligned}
v(Y_\alpha) &= \int_A V(D\alpha) \\
&= \int_{x\in A}\sqrt{1 + \sum_{i=1}^{k}(\partial_i f(x))^2}
\end{aligned}$$

□

Exercise 2.1. We are given the parametrization $\sigma : U \to \mathbb{R}^3$ of an orientable surface, where $U = (0,1) \times (0,\pi)$ and σ is defined by $\sigma(u,v) = (u\cos v, u\sin v, v)$, for all $(u,v) \in U$. Since U is open in $\mathbb{R}^2$, and σ is $\mathcal{C}^1$, then σ is a parametrized 2-manifold in $\mathbb{R}^3$. Now,

$$D\sigma(u,v) = \begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \\ 0 & 1 \end{pmatrix}.$$

Hence, by a formula derived in lecture, the area $\mathcal{A}(\sigma(U))$ is given by:

$$\begin{aligned}
A(\sigma(U)) &= \int_U |\det(D\sigma)^T(D\sigma)|^{1/2} \\
&= \int_U \left| \begin{pmatrix} \cos v & -\sin v & 0 \\ -u\sin v & u\cos v & 1 \end{pmatrix} \begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \\ 0 & 1 \end{pmatrix} \right|^{1/2} \\
&= \int_U \left| \begin{pmatrix} 1 & 0 \\ 0 & u^2+1 \end{pmatrix} \right|^{1/2} \\
&= \int_{(0,1)\times(0,\pi)} \sqrt{u^2+1} \\
&= \int_0^\pi \int_0^1 \sqrt{u^2+1} && \text{by Fubini.} \\
&= \pi \int_0^{\pi/4} \sec^3\theta d\theta && \text{by the substitution } \tan\theta = u \\
&= \pi \left(\tan\theta\sec\theta|_0^{\pi/4} - \int_0^{\pi/4} (\sec^2\theta - 1)\sec\theta d\theta \right) && \text{int. by parts with} \\
& && u = \sec\theta, dv = \sec^2\theta
\end{aligned}$$

$$\begin{aligned}
\implies 2\int_0^{\pi/4} \sec^3\theta d\theta &= \tan\theta\sec\theta|_0^{\pi/4} + \int_0^{\pi/4} \sec\theta d\theta \\
\int_0^{\pi/4} \sec^3\theta d\theta &= \frac{\sqrt{2}}{2} - 0 + \frac{1}{2}\ln(1+\sqrt{2}) - \frac{1}{2}\ln(1) \\
&= \frac{1}{\sqrt{2}} + \frac{1}{2}\ln(1+\sqrt{2}).
\end{aligned}$$

Hence, the area is given by:

$$A(\sigma(U)) = \frac{\pi}{\sqrt{2}} + \frac{\pi}{2}\ln(1+\sqrt{2}).$$

$\square$