looking for ... basis

$\mathbb{R} \to$ scalars

Then: the basis of $W_1 \cap P_n(\mathbb{R})$:

is $\{1, x^2, \ldots, x^{2k_1}\}$, where $2k_1 \leq n$ & $2(k+1) > n$

So if n = even $\Rightarrow \dim(W_1 \cap P_n(\mathbb{R})) = \frac{n}{2} + 1$

if n = odd $\Rightarrow \dim(W_1 \cap P_n(\mathbb{R})) = \frac{n+1}{2} + 1$

* the basis of $W_2 \cap P_n(\mathbb{R})$: $\{x, x^3, \ldots, x^{2k_2+1}\}$, $2k_2 + 1 \leq n$
$2(k_2+1) + 1 > n$

So if n = even $\Rightarrow \dim(W_2 \cap P_n(\mathbb{R})) = \frac{n}{2}$
if n = odd $\Rightarrow \dim(W_2 \cap P_n(\mathbb{R})) = \frac{n-1}{2}$

Linear Transformations

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17.10.06

... but first

1. (fishy) Theorem: Every vector space has a basis.

Examples $\mathbb{R}/\mathbb{Q}$ a: $\{1, \sqrt{2}, \pi, e, \ldots$? we don't know if $\pi + e$ are rational
$\pi + e = r.1$
$e = r.1 - \pi$

2. Lagrange Approximation

Let $(x_i)_{i=1}^{n+1} = (x_1, \ldots x_{n+1})$ be distinct numbers in $\mathbb{R}$

Let $(y_i)_{i=1}^{n+1}$ be arbitrary numbers in $\mathbb{R}$.

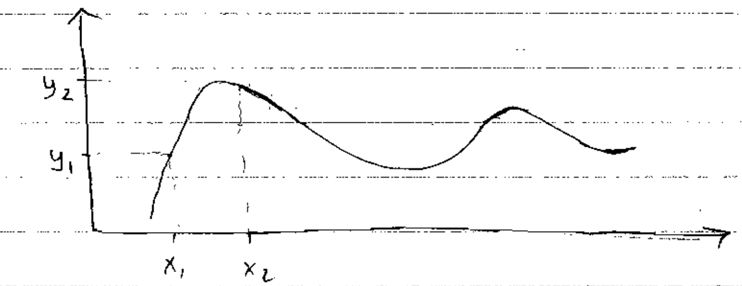
Question: Can you find a polynomial $p \in P_n(\mathbb{R})$ s.t. $P(x_i) = y_i$?

Is it unique?

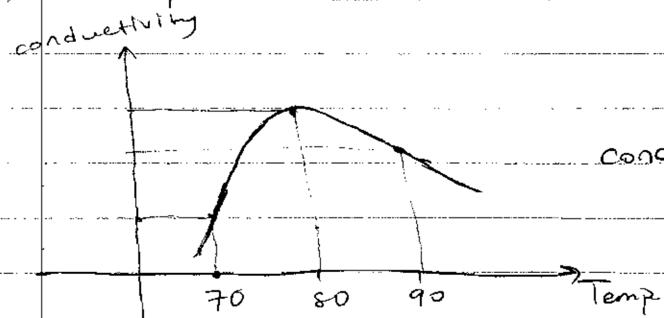
1. Who cares?

a. Computer drawing programs

a)



b) All experimental scientists.

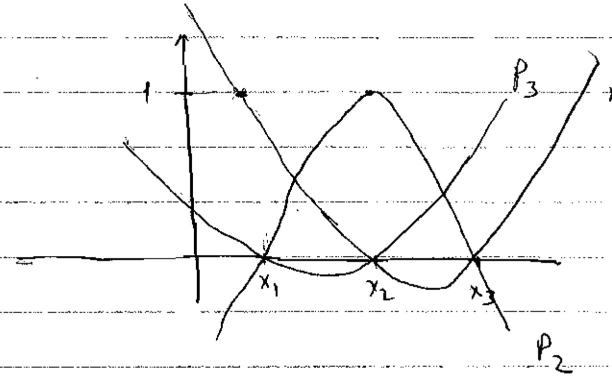


$\text{cond}(T) = f(T)$

Answer: Yes

Assume you can find polynomials $P_j \in P_n(\mathbb{R})$ s.t.

$$P_j(x_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$



P_1: $x_1 \longrightarrow 1$, $x_2 \longrightarrow 0$, $x_3 \longrightarrow 0$, $\vdots$

P_2: $x_1 \longrightarrow 0$, $x_2 \longrightarrow 1$, $x_3 \longrightarrow 0$, $\vdots$

Existence:

$$P = \sum_{j=1}^{n+1} y_j P_j$$ ① This is a lin. comb. of the P_j's, so it is in $P_n(R)$

② $P(x_i) = \sum y_j P_j(x_i) = y_i P_i(x_i) = y_i \cdot 1 = y_i$

all terms are zero except if $i=j$

Uniqueness:

Claim: The P_j's form a basis of $P_n(R)$

Prf: They are lin. indep.; indeed, assume

$$\sum a_j P_j = 0 \xrightarrow[x = x_i]{\text{substitude}} \sum a_j P_j(x_j) = 0 \Rightarrow a_i = 0$$

$\Rightarrow \forall i,\ a_i = 0$

An easy basis is $\underbrace{(x^0, x^1, x^2, \ldots, x^n)}_{n+1}$

$\dim P_n(\mathbb{R}) = n+1$

The set $\{P_i\}_{j=1}^{n+1}$ has $n+1$ elements and it is lin. indep. so it is a basis.

Assume p satisfies $p(x_i) = y_i$ as the P_j's form a basis, $\exists$ unique b_j's

s.t. $p = \sum b_j P_j$

Substitude x_i,

$y_i = P(x_i) = \sum b_j P_j(x_i) = b_j$, so $b_i = y_i$, and the curren P is

equal the old one. $\square$

It remains to find ~~the~~ a collection of P_j's

Lagrange Formula:

because $(x_j - x_j) = 0$ so I drop

$$P_j(x) = \frac{(x-x_1)(x-x_2)\cdots\cancel{(x-x_j)}\,(x-x_{j+1})\cdots(x-x_{n+1})}{\text{Dror}}$$

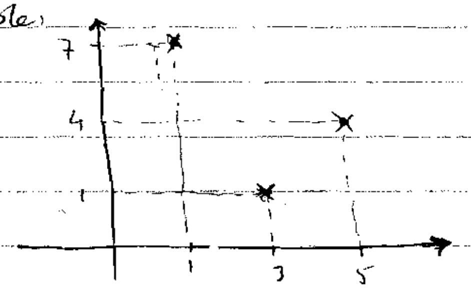
$$= \frac{\prod_{i\neq j}(x-x_i)}{\text{Dror}} = \frac{\prod_{i\neq j}(x-x_i)}{\prod_{i\neq}(x_j-x_i)}$$

If $i \neq j$

$$P_j(x_i) = \frac{(x_i-x_i)\cdots(x_j-x_{n+1})}{\text{Dror}} = 0$$

$$1 \overset{?}{=} P_j(x_j) = \frac{\overset{\neq 0}{(x_j-x_1)}\overset{\neq 0}{(x_j-x_2)}\cdots\cancel{(x_j-x_j)}\cdots(x_j-x_{n+1})}{\text{Dror}} = \frac{\text{Dror}}{\text{Dror}}$$

Example:



in $P_2(\mathbb{R})$

$$P_1 = \frac{\prod_{i\neq 1}(x-x_i)}{\prod_{i\neq 1}(x_1-x_i)} = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$= \frac{(x-3)(x-5)}{(1-3)(1-5)} = \frac{1}{8}(x-3)(x-5) = \frac{1}{8}(x^2-8x+15)$$

Dror

$p(1)=7 \quad x_1=7 \quad y_1=7$

$p(3)=1 \quad x_2=3 \quad y_2=1$

$p(5)=4 \quad x_3=5 \quad y_3=4$

likewise we can compute P_2, P_3

$$P_2 = \frac{(x-1)(x-5)}{(3-1)(3-5)} = -\frac{1}{4}(x^2-6x+5)$$

$$P_3 = \frac{(x-1)(x-3)}{(5-1)(5-3)} = \frac{1}{8}(x^2-4x+3)$$

$P_j(x_i) = 0$ $\quad P_3(x_2) = \frac{1}{8}(3^2-4\cdot 3+3) = 0$ ✓

$i \neq j$

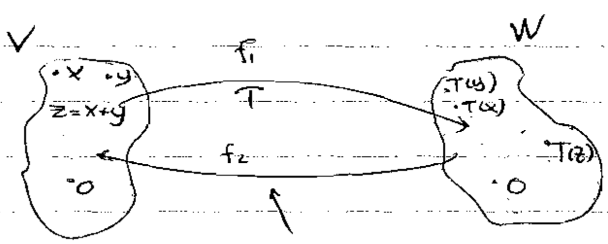
$P_j(x_j) = 1$ $\quad P_3(x_3) = \frac{1}{8}(5^2-4\cdot 5+3) = \frac{8}{8} = 1$ ✓

$$P = \sum y_j P_j = 7P_1 + 1P_2 + 4P_3$$

$$= 7\cdot\frac{1}{8}(x^2-8x+15) + \cdots$$

Ex: P works $\quad P(3)=1 \quad \cdots$

$\qquad P(1)=7 \quad \cdots$



structure $\qquad$ respect the structure $\qquad$ structure

If two structures are the same

student $\longrightarrow$ student number

isomorphism.

$$T(z) = T(x) + T(y)$$

$$T(z) = T(x+y) = T(x) + T(y)$$

Def: let V and W be v.s. over the same field F.

A "linear transformation" $\overset{T}{\downarrow}$ from V to W is a function $T: V \longrightarrow W$ s.t.

"T respect structure" ① $T(0) = 0$

② $\forall x, y \in V \quad T(x+y) = T(x) + T(y)$

③ $\forall c \in F \quad \underbrace{T(\overbrace{cx}^{\text{in } V})}_{\text{in } W} = \underbrace{c\,T(\overbrace{x}^{\text{in } V})}_{\text{in } W}$

Examples:

$$V = \mathbb{R}^2 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \right\} \quad W = M_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\}$$

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a-b & 7b \\ 2a & 2a+3b \end{pmatrix}$$

Claim: T is linear

1. $T(0_V) = T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_W$

2. $T\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right) \overset{?}{=} T\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + T\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

$$= T\begin{pmatrix} a_1+a_2 \\ b_1+b_2 \end{pmatrix} = \begin{pmatrix} 3(a_1+a_2) & 7(b_1+b_2) \\ \_ & \_ \end{pmatrix} \overset{?}{=} \begin{pmatrix} 3a_1 - b_1 & 7b_1 \\ \_ & \_ \end{pmatrix} + \begin{pmatrix} 3a_2 - b_2 & 7b_2 \\ \_ & \_ \end{pmatrix}$$

$$= \begin{pmatrix} 3a_1 - b_1 + 3a_2 - b_2 & 7b_1 + 7b_2 \\ \_ & \_ \end{pmatrix}$$

$$= \begin{pmatrix} \overbrace{3a_1+3a_2-b_1-b_2}^{\text{first term opened}} & 7b_1+7b_2 \\ - & - \end{pmatrix} \checkmark$$

3. exerxise

Example

$T: P_n(\mathbb{R}) \longrightarrow P_{n-1}(\mathbb{R})$ Linear Trans.

$TP = P'$

1. $T(0) = 0$? $\Rightarrow 0' = 0$!

2. $T(p+q) = T(p) + T(q)$

$(p+q)' = p' + q'$

3. $(cf)' = cf'$

Non-example.

$T: \mathbb{R}^1 \longrightarrow \mathbb{R}^1$

$x \longrightarrow x^2$

1. $T(0) = 0^2 = 0$

2. $T(x+y) = (x+y)^2 \neq x^2+y^2 = T(x)+T(y)$ X