12.10.06 Lecture

Claim:

Every lin. independent set L in a finite-dimentional vector space V can be extended to a basis of V.

Prf:

1. Add elements to L, one by one, making sure that each additional element is not in the span of what you had before, the result remains lin. indep

2. $\hookrightarrow$ take G to be some basis of V, use replacement.

get $H \subset G$ s.t. $|H| = |G| - |L|$ & $H \cup L$ generates and is extension of L

$|H \cup L| \leq |H| + |L| = |G| - |L| + |L| = |G| = \underbrace{n}_{\dim V}$

but $|H \cup L| \geq n \Rightarrow |H \cup L| = n$

so $H \cup L$ is a basis $\square$

Thm: Assume V is finite dimentional (f.d.) and $W \subset V$ is a subspace.

1. W is also f.d.

2. $\dim W \leq \dim V$

3. Every basis of W can be extended to be a basis of V

Prf: ① Pick elements of W one by one, making sure that no one is in the span of the prev. ones. Choose $w_1 \in W$ s.t. $w_1 \neq 0$ (if impossible, $W = \{0\}$, $\emptyset$ is a bas

Choose $w_2 \notin \operatorname{span}(w_1)$ (If impossible, $\{w_1\}$ is a basis)

(w_2 in W)

Assume $w_1 \dots w_k$ were chosen in W s.t. $(w_1 \dots w_k)$ is lin. indep.

If generates, it is a basis and we are done. If not, there exist w_{k+1}

$w_{k+1} \notin \operatorname{span}(w_1 \dots w_k)$ so $w_1 \dots w_{k+1}$ is still lin. indep. and we ~~can~~ continue. The process is guaranteed to stop for some $k \leq n$ as every linearly indep. set in V has at most n elements.

② Any basis α of W is lin. indep. in V so $\underbrace{|\alpha|}_{= \dim W} \leq \dim V$

③ If $\alpha \subset W$ is a basis, it is lin. indep. hence by previous claim it can be extended.

Thm: (Dror's Opinion: Fishy) Every vector space has a basis.

Prf uses "the axiom of choice"