Suppose M is finitely generated w/ generators $g_1, \dots, g_n \in M$. Ie. M is an R-module,
$M = \left\{ \sum_{i=1}^{n} a_i g_i : a_i \in R \right\}$.

Consider the map $\pi : R^n \twoheadrightarrow M$, $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mapsto \sum_{i=1}^{n} a_i g_i$. π is clearly onto. Is π an R-module homo?

$\pi\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \pi\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} a_i g_i + \sum_{i=1}^{n} b_i g_i = \sum_{i=1}^{n} (a_i + b_i) g_i = \pi\begin{pmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{pmatrix}$

yes. the other is also obvious.

So, $R^n / \ker \pi \cong \operatorname{im}(\pi) = M$. Now, $\ker(\pi) = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in R^n : \sum_{i=1}^{n} a_i g_i = 0 \right\} \to$ doesn't tell us much.

Let $X = \ker(\pi) \subset R^n$. Let $\{x_\eta : \eta \in J\}$ be a set of generators over R of X, ie. $X = \langle x_\eta : \eta \in J \rangle =$
$\operatorname{span}_R(\{x_\eta : \eta \in J\}) = \left\{ \sum_{\eta \in J} r_\eta x_\eta : r_\eta \in R,\ r_\eta \neq 0 \text{ for at most finitely many } \eta\text{'s} \right\} = RX$

Let $R^X = \left\{ a : X \to R \mid a(x) \neq 0 \text{ for finitely many } x\text{'s} \right\}$. (Note: this is not the usual R^X, which is the set of fns from X to R.)
↳ Here, we're thinking of X as an index set.

can think of elements of R^X as "X-tuples": $a \in R^X \Leftrightarrow a = (\underset{\in R}{a(x_1)}, \underset{\in R}{a(x_2)}, \underset{\in R}{a(x_3)}, \dots) = (a_{x_1}, a_{x_2}, a_{x_3}, \dots)$

Define a map $A : R^X \to R^n$ by
$a \mapsto \sum_{\eta \in J} a(x_\eta) x_\eta$ (ok b/c only finitely many are nonzero.)

↳ can think of $\sum_{\eta \in J} a(x_\eta) x_\eta$ in terms of matrix mult.:

If $x_\eta = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, $a(x_\eta) x_\eta = a(x_\eta)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a(x_\eta) b_1 \\ \vdots \\ a(x_\eta) b_n \end{pmatrix}$. So, if $J = \{1, 2, \dots\}$, $\sum_{\eta \in J} a(x_\eta) x_\eta = \sum_{j=1}^{\infty} a(x_j) x_j = (a(x_1), a(x_2), \dots) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}$

So by construction, $\operatorname{im}(A) = X = \ker(\pi)$. Thus, $M \cong R^n / \ker(\pi) = R^n / \operatorname{im}(A) =: M_A$ ← module gen. by A over R.

$$\begin{array}{ccccc} R^X & \xrightarrow{A} & R^n & \xrightarrow{\pi} & M \\ a & \mapsto & \sum_{\eta \in J} a(x_\eta) x_\eta & & \\ & & \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} & \mapsto & \sum_{i=1}^{n} b_i g_i \end{array}$$

(Easy to check: A is an R-mod morphism)

where $X = \langle x_\eta : \eta \in J \rangle \subset R^n$.

If $J = \{1, 2, \dots\}$, think of e_{x_j} as an infinite basis vector: $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$ ← j^{th} spot

$\forall \eta \in J$, let $e_{x_\eta} \in R^X$ be the function $e_{x_\eta}(y) = \begin{cases} 0 & \text{if } y \neq x_\eta \\ 1 & \text{if } y = x_\eta \end{cases}$.

Notice that $A(e_{x_\eta}) = \sum_{\nu \in J} e_{x_\eta}(x_\nu) x_\nu = x_\eta$, and also $\operatorname{im}(A) = \langle A(e_{x_\eta}) : \eta \in J \rangle$, since
$\forall a \in R^X$, $A(a) = \sum_{\eta \in J} a(x_\eta) x_\eta = \sum_{\eta \in J} \underbrace{a(x_\eta)}_{\text{coordinates in } R} \underbrace{A(e_{x_\eta})}_{\text{"basis vectors for } X\text{"}} \left(= A\left(\sum_{\eta \in J} a(x_\eta) e_{x_\eta} \right) \right)$

Can think of $\operatorname{im}(A)$ as a $|J| \times 1$ column vector wrt the "basis" $\{e_\eta : \eta \in J\}$ of X