MATH 240 – FALL 2014

# HOMEWORK ASSIGNMENT #3

CORRECTION

## Algebra I

08/10/2014

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**Exercise 1 page 32:** *Label the following statement as true or false.*

| Statements | | Label | Comments |
|---|---|---|---|
| (a) | The zero vector is a linear combination of any nonempty set of vectors | True | $A \neq \emptyset$ and $A \subset V \Longrightarrow \exists x \in A$. Moreover we have $0 = 0 \cdot x$. |
| (b) | The span of $\emptyset$ is $\emptyset$ | False | $Span(\emptyset) = \{0\}$ |
| (c) | If $S$ is a subset of a vector space $V$, then $Span(S)$ equals the intersection of all subspaces of $V$ that contain $S$ | True | $Span(S)$ is a subspace $\wedge\, S \subset Span\,(S) \Longrightarrow \bigcap_{U \in \mathcal{S}} U \subset Span(S)$ where $\mathcal{S}$ is the set of all subspace $U$ of $V$ s.t. $S \subset U$. Conversely, if $v \in Span(S)$, $v$ is a linear combination of elements of $S$ so $v \in U\ \forall U \in \mathcal{S}$ since $U$ is closed under addition and scalar multiplication $\Longrightarrow v \in \bigcap_{U \in \mathcal{S}} U \quad \Longrightarrow Span(S) \subset \bigcap_{U \in \mathcal{S}} U.$ Hence $Span(S) = \bigcap_{U \in \mathcal{S}} U$ |
| (d) | In solving a system of linear equations, it is permissible to multiply an equation by any constant | False | You can not multiply an equation by $c = 0$ since it will change the solutions of the system of linear equations |
| (e) | In solving a system of linear equations, it is permissible to add any multiple of one equation to another | True | |
| (f) | Every system of linear equations has a solution | False | $\begin{pmatrix}1\\0\end{pmatrix} = a\begin{pmatrix}0\\1\end{pmatrix}$ does not have a solution |

**Exercise 2. (f) page 33.** *Solve the following systems of linear equations by the method introduced in this section*

The system (f) of linear equation is the following:

$$\begin{array}{rcrl} x_1 + 2x_2 + 6x_3 &=& -1 & e_1 \\ 2x_1 + x_2 + x_3 &=& 8 & e_2 \\ 3x_1 + x_2 - x_3 &=& 15 & e_3 \\ x_1 + 3x_2 + 10x_3 &=& -5 & e_4 \end{array} \quad \rightarrow \quad \begin{array}{rcrl} x_1 + 2x_2 + 6x_3 &=& -1 & e_1 \\ -3x_2 - 11x_3 &=& 10 & e_2 - 2e_1 \\ -5x_2 - 19x_3 &=& 18 & e_3 - 3e_1 \\ x_2 + 4x_3 &=& -4 & e_4 - e_1 \end{array}$$

$$\begin{array}{rcrl} x_1 + \quad - 2x_3 &=& 7 & e_1 - 2e_4 \\ x_3 &=& -2 & e_3 + 3e_4 \\ x_3 &=& -2 & e_3 + 5e_4 \\ x_2 + 4x_3 &=& -4 & e_4 \end{array} \quad \rightarrow \quad \begin{array}{rcrl} x_1 + \quad - 2x_3 &=& 7 & e_1 \\ x_2 + 4x_3 &=& -4 & e_4 \\ x_3 &=& -2 & e_3 \\ x_3 &=& -2 & e_2 \end{array}$$

$$\begin{array}{rcrl} x_1 &=& 3 & e_1 + 2e_2 \\ x_2 &=& 4 & e_2 - 4e_3 \\ x_3 &=& -2 & e_3 \\ 0 &=& 0 & e_4 - e_3 \end{array}$$

So finally $x_1 = 3, x_2 = 4$ and $x_3 = -2$.

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**Exercise 3. (f) page 33.** *For each of the following list of vectors in $\mathbb{R}^3$, determine whether the first vector can be expressed as a linear combination of the other two.*

We are looking for $a, b$ such that:

$$\begin{pmatrix}-2\\2\\2\end{pmatrix} = a \cdot \begin{pmatrix}1\\2\\-1\end{pmatrix} + b \cdot \begin{pmatrix}-3\\-3\\3\end{pmatrix} = \begin{pmatrix}a-3b\\2a-3b\\-a+3b\end{pmatrix}$$

Moreover, 2 vectors are equals if their coordinates are equal. Thus we get the following system of equations:

$$\begin{array}{rl}a-3b&=-2\\2a-3b&=2\\-a+3b&=2\end{array}\quad\begin{array}{l}e_1\\e_2\\e_3\end{array}\quad\rightarrow\quad\begin{array}{rl}a-3b&=-2\\a&=4\\0&=0\end{array}\quad\begin{array}{l}e_1\\e_2-e_1\\e_3+e_1\end{array}$$

$$\begin{array}{rl}-3b&=-6\\a&=4\\0&=0\end{array}\quad\begin{array}{l}e_1-e_2\\e_2\\e_3\end{array}\quad\rightarrow\quad\begin{array}{rl}a&=4\\b&=2\\0&=0\end{array}\quad\begin{array}{l}e_2\\e_1/-3\\e_3\end{array}$$

So finally $a = 4, b = 2$ so we have:

$$\begin{pmatrix}-2\\2\\2\end{pmatrix} = \begin{pmatrix}4-3\cdot 2\\2\cdot 4-3\cdot 2\\-4+3\cdot 2\end{pmatrix}$$

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**Exercise 4.(f) page 32**: *for each list of polynomials in $P_3(\mathbb{R})$, determine whether the first polynomial can be expressed as a linear combination of the other two.*

We are looking for $a, b$ such that:

$$6x^3 - 3x^2 + x + 2 = a \cdot (x^3 - 2x^2 + 2x + 3) + b \cdot (2x^3 - 3x + 1)$$

So we get:

$$6x^3 - 3x^2 + x + 2 = (a + 2b)x^3 + (-2b)x^2 + (2a - 3b) + 3a + b$$

Moreover, 2 polynomials are equals if all of their coefficients are equals so we have the following system of equations:

$$\begin{array}{rl}a+2b&=6\\-a&=-3\\2a-3b&=1\\3a+b&=2\end{array}\quad\begin{array}{l}e_1\\e_2\\e_3\\e_4\end{array}\quad\rightarrow\quad\begin{array}{rl}a+2b&=6\\a&=3\\2a-3b&=1\\3a+b&=2\end{array}\quad\begin{array}{l}e_1\\-e_2\\e_3\\e_4\end{array}$$

$$\begin{array}{rl}2b&=3\\a&=3\\-3b&=-5\\b&=-7\end{array}\quad\begin{array}{l}e_1-e_2\\e_2\\e_3-2e_2\\e_4-3e_2\end{array}\quad\rightarrow\quad\begin{array}{rl}0&=17\\a&=3\\-3b&=-5\\b&=-7\end{array}\quad\begin{array}{l}e_1-2e_4\\e_2\\e_3\\e_4\end{array}$$

So here we have got an inconsistent equation $e_1: 0 = 17$ so the system has no solutions.

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**Exercise 5.(h) page 34**: *In each part determine whether the given vector is in the span of* $S$.

Let us consider the $2 \times 2$ matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the following set $S$:

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is in the $Span(S)$ if and only if it is a linear combination of vectors in $S$. So it means that there exists $a, b, c \in F$ such that:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a \cdot \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

This become:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ -a & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a + c & c + b \\ -a & b \end{pmatrix}$$

Moreover, 2 matrices are equals if all of their coefficients are equals. Therefore we clearly see that $a$ must be equals to $0$ and $b = 1$ but this implies that $1 = a + c = c$. But the last coefficient gives us $c + b = 1 + 1 = 0$ which is inconsistent if we consider the vector spaces of $2 \times 2$ matrices over the real numbers $\mathbb{R}$. Therefore, there does not exist a linear combination of vectors of $S$ which gives us $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin Span(S)$.

**Exercise 10 page 34**: *Show that if*

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

*then the span of* $\{M_1, M_2, M_3\}$ *is the set of all symmetric* $2 \times 2$ *matrices*

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Let us consider the vector space $\mathcal{M}_{2\times2}(F)$.
Let:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $S = \{M_1, M_2, M_3\}$. Let us consider any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times2}(F)$ where $a, b, c, d \in F$ such that $M \in Span(S)$. By definition, the span of $S$ is the set of all linear combinations of elements of $S$. Therefore, $\exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \alpha_2 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_3 \\ \alpha_3 & 0 \end{pmatrix}$$

This gives us:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_1 \end{pmatrix}$$

Moreover, 2 matrices are equals if all of their coefficients are equals. So we have $a = \alpha_1$, $b = \alpha_3 = c$ and $d = \alpha_1$

Therefore, if $M \in Span(S), M$ must be of the form $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$. Thus we have:

$$Span(S) = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \mid a, b, d \in \mathbb{R} \right\}$$

Hence $Span(S)$ is the set of all $2 \times 2$ symmetric matrices.

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