Oct 12, 2012 Mat267 1/2

Theorem

Given $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ sufficiently differentiable,
if $P_0 \in \mathbb{R}^n$, $g(P_0) = 0$, $\nabla g(P_0) \neq 0$ and $\nabla f(P_0) \not\propto \nabla g(P_0)$. (not proportional)

$\Leftrightarrow \nexists \lambda$ s.t. $\nabla f = \lambda \nabla g$

$\Leftrightarrow \nexists \lambda$ s.t. $\nabla(f + \lambda g) = 0$

then arbitrarily near P_0 can find $P_\pm$ s.t. $g(P_\pm) = 0$ & $f(P) < f(P_0) < f(P_\pm)$

proof (sketch)

Find v s.t. $v \perp \nabla g(P_0)$ yet $v \not\perp \nabla f(P_0)$. Then

$$D_{P_0, v}(f) = \frac{d}{d\varepsilon} f(P_0 + \varepsilon v)\Big|_{\varepsilon = 0}$$
$$= \nabla f(P_0) \cdot v$$
$$> 0$$

$$D_{P_0 v}(g) = \frac{d}{d\varepsilon} g(P_0 + \varepsilon v)\Big|_{\varepsilon = 0}$$
$$= \nabla g(P_0) \cdot v$$
$$= 0$$

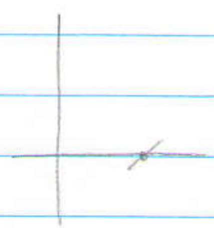
$\tilde{P}_\pm = P_0 \pm \varepsilon v$ for small ε, then $f(\tilde{P}_+) \underbrace{>}_{\sim\varepsilon} f(P_0) \underbrace{>}_{\sim\varepsilon} f(\tilde{P}_-)$

yet,

$$\left.\begin{aligned} |g(\tilde{P}_+) - g(P_0)| &\sim \varepsilon^2 \\ |g(\tilde{P}_-) - g(P_0)| &\sim \varepsilon^2 \end{aligned}\right\} \text{very small}$$

Hard claim

Near a near 0 of a function whose $\nabla g \neq 0$, you can find an actual 0.



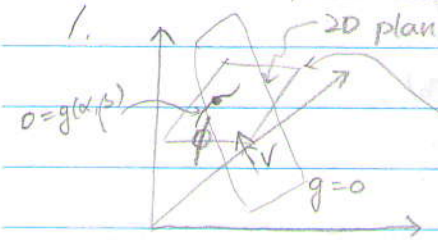
"Implicit Function Theorem"

$\rightsquigarrow$ Find points $P_\pm$, very near $\tilde{P}_\pm$ s.t

$g(P_\pm) = 0$ $|P_\pm - \tilde{P}_\pm| \sim \varepsilon^2$ $\square$

Two ways to relate it to Calculus of Variation.

1\.



$J(y) = \int F(x,y,y')\,dx$

$\{\phi + \alpha\psi_1 + \beta\psi_2\}$

$\to$ On V the ∞-problem reduces to the finite order

2\. In Calculus of Variation

$f \rightsquigarrow \int F(x,y,y')\,dx = J(y)$

$g \rightsquigarrow \int g(x,y,y')\,dx = K(y)$

$\nabla_p \rightsquigarrow EL_\phi(F) = (F_y - \frac{d}{dx}F_{y'})(\phi)$

Indeed,

$$D_v f = \frac{d}{dt} f(p+\varepsilon v)\Big|_{\varepsilon=0} = (\nabla_p f, v)$$

$$D_{\phi,h}(J) = \frac{d}{d\varepsilon} J(\phi+\varepsilon h)\Big|_{\varepsilon=0} = \cdots = \int_a^b (F_y - \tfrac{d}{dx}F_{y'})\,h\,dx$$

$\phi : [a,b] \to \mathbb{R}$
$h : [a,b] \to \mathbb{R}$
$h(a)=h(b)=0$

$$= \int EL_\phi(F)\cdot h\,dx = (EL_\phi(F), h)$$

inner product

$\underbrace{(V,W) = \sum V_i W_i}_{\text{finite dimension}} \rightsquigarrow \underbrace{(f,g) = \int f(x)g(x)\,dx}_{\text{infinite dim.}}$

$$\nabla(f+\lambda g) = 0 \rightsquigarrow EL_\phi(F+\lambda G) = 0$$

Numerical Methods

$y' = f(x,y) \qquad \phi(x_0) = y_0 \qquad \phi'(x) = f(x,\phi(x))$

1\. Use the proof of Picard's Theorem

$\phi_0(x) \equiv y_0$

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f(x, \phi_{n-1}(x))\,dx$$

$$\phi_n(x) \xrightarrow{\text{ugly}} \phi(x)$$

<u>example</u>

$y' = -y$, $\phi(0) = 1$

Find $\phi(100)$

$(\Rightarrow \phi(x) = e^{-x} \longrightarrow \phi(100) = e^{-100} \sim 0)$

$\phi_0 \equiv 1$

$\phi_1(x) = 1 - x$

$\phi_2(x) = 1 - x + \frac{x^2}{2}$

$\vdots$

$$\phi_n(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^n}{n!}$$

$\phi_0(100) = 1$

$\phi_1(100) = -99$

$\phi_2(100) = -99 + 5000 = 4901$

$\phi_3(100) \sim -10^6$