

$$\textcircled{2} \quad W \cap W' = \{0\}$$

Suppose that  $\exists w \neq 0, w \in W \cap W'$ , then  $w \in W, w \in W'$ .

Then  $\exists$  unique  $d_1, d_2, \dots, d_m, \gamma_1, \gamma_2, \dots, \gamma_n$ , s.t.

$$d_1 u_1 + d_2 u_2 + \dots + d_m u_m = w. \quad \textcircled{1}$$

$$\gamma_1 v_1 + \gamma_2 v_2 + \dots + \gamma_n v_n = w. \quad \textcircled{2}$$

$$\textcircled{1} - \textcircled{2} :$$

$$\sum_{i=1}^m d_i u_i - \sum_{i=1}^n \gamma_i v_i = 0. \quad \textcircled{3}$$

$\because w \neq 0, \beta_1, \beta_2$  are all linearly independent

$\therefore d_i, \gamma_i$  are not all 0.

Then there is a contradiction between  $\textcircled{3}$  and

$\beta$  is linearly independent.

Hence  $W \cap W' = \{0\}$

$\Rightarrow W + W' = V$ , particularly, if  $\beta_1 = \beta$ , then  $\beta_2 = \emptyset$ ,

We also have  $W' = \{0\}$  s.t.  $W = V, W + W' = V$

Then there must exist  $T: V \rightarrow V$  is a projection on  $W$  along  $W'$

s.t. if  $x \in V, x = x_1 + x_2, x_1 \in W, x_2 \in W'$ , then  $T(x) = x_1$

(b) Proof:

Suppose that:  $W_1, W_2 \subseteq V$ ,  $W_1 \cap W_2 = \{0\}$ ,  $W_1 \oplus W_2 = V$

$\beta_1 = \{v_1, v_2, \dots, v_m\}$ ,  $\beta_2 = \{u_1, u_2, \dots, u_n\}$  are bases of  $W_1, W_2$  respectively. Suppose  $\beta = \beta_1 \cup \beta_2$ , then  $\beta$  is a basis of  $V$  by problem 29 in Ex 1.6.

$\forall v \in W_1, \exists$  unique  $\alpha_1, \dots, \alpha_m \in F$  s.t  $v = \alpha_1 v_1 + \dots + \alpha_m v_m$

$\forall u \in W_2, \exists$  unique  $\gamma_1, \dots, \gamma_n \in F$  s.t  $u = \gamma_1 u_1 + \dots + \gamma_n u_n$

Then  $V = W_1 \oplus W_2 = \{v+u \mid v \in W_1, u \in W_2\}$

$$= \{\alpha_1 v_1 + \dots + \alpha_m v_m + \gamma_1 u_1 + \dots + \gamma_n u_n\}$$

Take a  $v_0 \neq 0 \in W_1$  and if  $u_i \in \beta_2$ , we can create

$\star \quad W_3 = \text{span}\{v_0 + u_i \mid v_0 \in W_1, u_i \in \beta_2\}$

Then by  $v_0 \neq 0, u_i \neq 0 \Rightarrow v_0 + u_i \neq v_0, v_0 + u_i \neq u_i$

$$\text{So } W_3 \subseteq ((W_1 + W_2) \setminus W_1) \cup \{0\}$$

Now we want to show that  
 $\dim W_3 = \dim W_2$

$\forall w \in W_3, \exists \delta_1, \dots, \delta_n \in F$  s.t.

$$w = \delta_1(v_0 + u_1) + \dots + \delta_n(v_0 + u_n)$$

$$= (\delta_1 + \dots + \delta_n)v_0 + \delta_1 u_1 + \dots + \delta_n u_n$$

for  $\beta_2 = \{u_1, \dots, u_n\}$  is linearly independent

then  $\delta_1 u_1 + \dots + \delta_n u_n = 0$  iff  $\delta_1 = \dots = \delta_n = 0 \Rightarrow w = 0$

$$w = 0 \Rightarrow (\delta_1 + \dots + \delta_n)v_0 + \delta_1 u_1 + \dots + \delta_n u_n = 0 \quad \Rightarrow v_0 \in W_3, \\ W_3 \cap W_1 = \{0\}$$

For  $v_0 \notin W_2$ , then  $v_0$  cannot be expressed as a linear combination of  $\beta_2$

$$\text{Then } \sum_{i=1}^n \alpha_i + \dots + \alpha_n = 0$$

$$\sum_{i=1}^n \alpha_i u_1 + \dots + \alpha_n u_n = 0$$

$$\Rightarrow \alpha_1 = \dots = \alpha_n = 0.$$

Then  $\{v_0 + u_1, \dots, v_0 + u_n\}$  is linearly independent and spans  $W_3$

So it's a basis of  $W_3$

$$\Rightarrow \dim W_3 = n = \dim W_2$$

$\forall v \in W_3, \forall v \in W_1$  Suppose that  $v_0 = \alpha_1 v_1 + \dots + \alpha_m v_m$

$$v + v_0 = (\alpha_1 + \dots + \alpha_n) v_0 + \alpha_1 u_1 + \dots + \alpha_n u_n + \alpha_1 v_1 + \dots + \alpha_m v_m$$

$$= (\alpha_1 + \dots + \alpha_n)(\alpha_1 v_1 + \dots + \alpha_m v_m) + \sum_{i=1}^n \alpha_i u_i + \sum_{i=1}^m \alpha_i v_i$$

$$= \sum_{i=1}^n (\alpha_i + \alpha_n) \alpha_i v_i + \sum_{i=1}^n \alpha_i u_i + \sum_{i=1}^m \alpha_i v_i = \sum_{i=1}^n \alpha_i u_i + \sum_{i=1}^m (\alpha_i + \alpha_n) \alpha_i v_i$$

$$= u + v. \quad \forall u \in W_2, \forall v \in W_1$$

$$\Rightarrow W_1 \oplus W_2 = V$$

$w \in W_3$  and  $w \in W_2$  iff  $(\alpha_1 + \dots + \alpha_n) v_0 = 0$ .

$$\Rightarrow \alpha_1 + \dots + \alpha_n = 0 \text{ i.e. } \alpha_1 u_1 + \dots + \alpha_n u_n \text{ cannot span } W_2$$

for  $\alpha_1, \dots, \alpha_n$  are not arbitrary.

So  $W_3 \neq W_2$

Hence there can be 2 projections  $W_1$  along  $W_2$  &  $W_1$  along  $W_3$ .

Example  $V = \text{Span}(\beta)$ ,  $\beta = \{v_1, v_2, \dots, v_m, u_1, \dots, u_n\}$

$$W = \text{Span}(\beta_1), \quad \beta_1 = \{v_1, v_2, \dots, v_m\}$$

$$W_1 = \text{Span}(\beta_2), \quad \beta_2 = \{u_1, \dots, u_n\}$$

$$W_2 = \text{Span}(\beta_3), \quad \beta_3 = \{v_0 + u_1, \dots, v_0 + u_n\} \quad v_0 \in W \text{ & } v_0 \neq 0.$$