warning $|A+B| \neq |A|+|B|$

Pf If $k=1$ done otherwise $k \geq 2$

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} |A_{\hat{1}\hat{j}}|$$

row $k \rightsquigarrow$ row $k-1$

By induction $|A_{\hat{1}\hat{j}}|$ is linear in its row $k-1$, meaning in row k of A. So $\det(A)$ is a lin comb of lin. functions, hence it is linear in row k

Step 3 $\det(A)$ vanishes if the first two rows of A are equal.

$$\det\begin{pmatrix} \text{—} r \text{—} \\ \text{—} r \text{—} \\ \text{———} \\ \text{———} \end{pmatrix} = 0.$$

Pf assume $a_{1j} = a_{2j} = r_j$ then

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} |A_{\hat{1}\hat{j}}|$$

$$= \sum_{j=1}^{n} (-1)^{1+j} r_j$$

$$= \sum_{j=1}^{n} (-1)^{1+j} r_j \Big(\sum_{i<j} (-1)^{1+i} a_{2i} |A_{\hat{1}\hat{2},\hat{j}\hat{i}}|$$

$$+ \sum_{i>j} (-1)^{1+(i-1)} a_{2i} |A_{\hat{1}\hat{2},\hat{j}\hat{i}}| \Big)$$

$$= \sum_{j=1}^{n} (-1)^{1+j} r_j \sum_{i\neq j} r_i |A_{\hat{1}\hat{2},\hat{j}\hat{i}}| (-1)^{1+i} \begin{cases} +1 & i<j \\ -1 & i>j \end{cases},$$

$$= \sum_{\substack{i=1 \\ j=1 \\ i\neq j}}^{n} (-1)^{i+j} r_i r_j |A_{\hat{1}\hat{2},\hat{j}\hat{i}}| \cdot \begin{cases} +1 & i<j \\ -1 & i>j \end{cases}$$

Here the (i,j) term cancels the (j,i) term so we get 0

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step 4 $\det(A)$ vanishes if any 2 adjacent rows in A are equal:

$$\det\begin{pmatrix} \text{———} \\ \text{———} \\ \text{— } r \text{ —} \\ \text{— } r \text{ —} \\ \text{———} \end{pmatrix} = 0$$

PF. Repeatedly use the co-factor expansion until the two r rows pop to the top, then we use step 3.

Problem for $A \in M_{n\times n}(F)$, compute

$$A^p = \underbrace{A \cdot A \cdots A}_{p\ \text{times}}$$

Example

what's $\begin{pmatrix} 4 & 3 \\ -6 & -5 \end{pmatrix}^{15}$?