MAT401 March 26 2008.

$\phi: R \to S$ $\qquad R/\ker\phi \cong \mathrm{im}\,\phi$

$\downarrow$

$\ker\phi$

The Plan:

$\{\text{field extensions}\} \xleftrightarrow{\text{Fundamental Theorem}} \{\text{Grays}\}$

$\{\text{extensions by radicals}\} \longrightarrow \{\text{solvable groups}\}$

$S_{\mathbb{Q}}(3x^5 - 15x + 5) \longrightarrow S_5$, not solvable

The 3 isomorphism theorems

1. $\phi: G_1 \longrightarrow G$ is a group homomorphism.
   $\ker\phi \triangleleft G_1$ & $G_1/\ker\phi \cong \mathrm{im}\,\phi$

2. If $N \triangleleft H \triangleleft G$ & $N \triangleleft G \Rightarrow G/H \cong {}^{G/N}\!/_{H/N}$

3. $H < G$, $N \triangleleft G \Rightarrow H/N\cap H \cong H\cdot N/N$

Partial proof of

$$\frac{H}{N \cap H} \cong \frac{H \cdot N}{N}$$

$$\| $$

$$\{[h] : h \in H\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \{[h \cdot n]_2 : h \in H, n \in N\}$$

$h_1 \sim h_2$ if $\qquad h_1 \cdot n_1 \overset{2}{\sim} h_2 \cdot n_2$ if

$h_1 = h_2 \cdot n$ where $\qquad h_1 n_1 = h_2 n_2 \cdot n$

$n \in N \cap H$ $\qquad$ for $n \in N$.

$\phi: [h]_1 \longrightarrow [h \cdot 1]_2$

claim: if $h_1 \overset{1}{\sim} h_2$ then $h_1 \overset{2}{\sim} h_2$, so ϕ is well-def $\square$

$\Psi : [h \cdot n]_2 \longmapsto [h]_1$

claim: if $h_1 n_1 \overset{2}{\sim} h_2 n_2$ then $h_1 \overset{1}{\sim} h_2$

Proof: $h_1 n_1 \overset{2}{\sim} h_2 n_2 \Rightarrow$ for some $n \in N$ $h_1 n_1 = h_2 n_2 \cdot n$

$\Rightarrow h_1 = h_2 \underbrace{n_2 \cdot n \cdot n_1^{-1}}_{\uparrow 2} \Rightarrow n_2 \cdot n \cdot n_1^{-1} \in N$ trivial

$n_2 n n_1^{-1} = h_2^{-1} h_1 \in H$

$N \cap H$

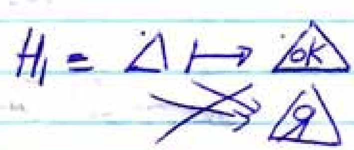
Def: A group G is called "solvable" if there exists a sequence of subgroups as follows.

$$G = H_k \triangleright H_{k-1} \triangleright H_{k-2} \triangleright \dots \triangleright H_1 \triangleright H_0 = \{0\}$$

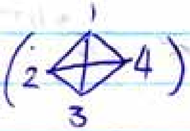
s.t. for each i, H_i / H_{i-1} is abelian.

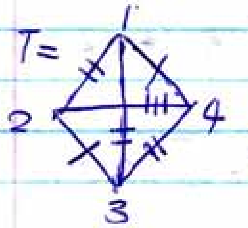
Examples:

1. Every Abelian group is solvable $G = H_1 \triangleright H_0 = \{e\}$
2. S_2 is solvable as it is abelian.
3. $S_3 = S(\triangle)$ is solvable
4. S_4

$H_1 =$ 

Example 4:

$G = S_4 = S($  $)$ is solvable.

$T =$ 

σ is in S_4 (Symmetry of T)
let $\phi(\sigma)$ be the induced permutation of opposite pairs; $\phi(\sigma) \in S_3$

$\phi : S_4 \longrightarrow S_3$

Example : $\phi\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix}$

pair $\mathrm{I} = \{\{1,4\},\{2,3\}\} = \{\{2,3\},\{4,1\}\} \longrightarrow \{\{2,1\},\{3,4\}\} = P\text{-}\mathrm{II}$
pair $\mathrm{II} = \{\{1,2\},\{3,4\}\} \longrightarrow \{\{2,3\},\{4,1\}\} = P\text{-}\mathrm{I}$
pair $\mathrm{III} = \{\{1,3\},\{2,4\}\} \longrightarrow \{\{2,4\},\{3,1\}\} = P\text{-}\mathrm{III}$

$N = \ker\phi = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 \end{pmatrix} \right\}$

$= \mathbb{Z}/2 \oplus \mathbb{Z}/2$ Abelian $\Rightarrow$ Solvable.

$G/N \cong S_3$ Solvable $\quad |G| = |G/N|\,|N|$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad 24 \quad\; 6 \quad 4$

Example 5 for $n \geq 5$ S_n is not solvable.

Proof:

$A_n =$ minimal subgroup of S_n containing all 3-cycles $\begin{pmatrix} 1 & 2 & i & j & k & n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & j & k & i & n \end{pmatrix}$

lemma 1: $[A_n, A_n] = A_n = (i,j,k)$

Proof: consider for $i \neq j \neq k \neq l \neq m$ $(n \geq 5)$,

$[(i,j,k)(k\,l\,m)] = (ijk)(klm)(ijk)^{-1}(klm)^{-1}$

$= (ijk)(klm)(jik)(mlk)$

$= \begin{pmatrix} 1 \cdots i & j & k & l & m & n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ l & j & i & k & m & \end{pmatrix}$

$= (i\,l\,k) \Rightarrow [A, A]$ contains all 3 cycles $\square$

lemma 1+2 prove example 5. Indeed, assume S_n was solvable.

$S^n = H_K \rhd H_{K-1} \rhd H_{K-2} \quad \rhd H_1 \rhd H_0 = \{1\}$.
s.t H_i / H_{i-1} is abelian

$A_n < H_K \Rightarrow A_n = [A_n, A_n] < H_{K-1} \Rightarrow A_n < H_{K-2} \Rightarrow \dots \Rightarrow A_n < H_0$
but $A_n \neq \{e\}$ so $\Rightarrow\!\times\!\Leftarrow$

Proof of lemma 2.

let $a, b \in A$ consider $aba^{-1}b^{-1} \in [A, A]$.
In H''/H', $[aba^{-1}b^{-1}] = \underbrace{[a][b][a]^{-1}[b]^{-1}}_{\text{In an abelian group.}} = [e]$.

so $aba^{-1}b^{-1} \in H'$
so H' contains the minimal subgroup containing all commutators, ie. $H' \supset [A, A]$.

Thm 1: if $N < G$, then N & G/N Solvable $\Leftrightarrow$ G is solvable

$\Rightarrow$: Assume N & G/N are solvable.

$N = H_K \rhd H_{K-1} \rhd \dots \rhd H_1 \rhd H_0 = \{e\}$.

$G/N = M_\ell \rhd M_{\ell-1} \rhd \dots \rhd M_0 = \{N\} = [e]$

s.t. H_i / H_{i-1} are abelian & M_j / M_{j-1}

Consider
$\pi : G \to G/N$. $G \ni g \xmapsto{\pi} [g] \in G/N$ (onto)

Consider

$$G = \underbrace{\pi^{-1}(M_\ell) > \dots \pi^{-1}(M_2) > \pi^{-1}(M_1)}_{\text{need to treat}} > \pi^{-1}(M_0) = N = \underbrace{H_k \triangleright \dots H_2 \triangleright H_1 \triangleright H_0 = \{e\}}_{\text{nothing to prove}}$$

claim ① $\pi^{-1}(M_{j-1})$ is normal in $\pi^{-1}(M_j)$

② and $\pi^{-1}(M_j)/\pi^{-1}(M_{j-1})$ is abelian

① Assume $a \in \pi^{-1}(M_{j-1}) \rightsquigarrow [a] \in M_{j-1}$

$b \in \pi^{-1}(M_j) \rightsquigarrow [b] \in M_j$

$\pi(b^{-1}ab) = [b^{-1}ab] = [b^{-1}][a][b] \underset{M_{j-1} \triangleleft M_j}{\overset{as}{\in}} M_{j-1}$

$b^{-1}ab \in \pi^{-1}(M_{j-1}) \triangleright$

② $\pi / \pi^{-1}(M_j) = \pi_j : \pi^{-1}(M_j) \longrightarrow G/N$

$\operatorname{Im} \pi_j = \pi(\pi^{-1}(M_j)) = M_j < G/N$

$\ker \pi_j = N \Rightarrow$ first Iso Thm : $\dfrac{\pi^{-1}(M_j)}{N} \cong M_j$

$M_j / M_{j-1} \cong \dfrac{\pi^{-1}(M_j)/N}{\pi(M_{j-1})/N} \underset{\text{2nd Iso Thm}}{\cong} \pi^{-1}(M_j)/\pi^{-1}(M_{j-1})$

$\frac{1}{2} \Leftarrow$: $N \triangleleft G$, G solvable $\Rightarrow G/N$ is solvable

Proof:

$G = H_K \triangleright H_{K-1} \triangleright \dots H_j \dots \triangleright H_1 \triangleright H_0 = \{e\}$. H_K/H_{K-1} abelian

$G/N = \dots\dots\dots \dfrac{H_j \cdot N}{N} > \dfrac{H_{j-1}N}{N} \dots$

claim $\dfrac{H_{j-1}N}{N}$ is normal in $\dfrac{H_jN}{N}$ & their quotient is abelian

Take $a \in \dfrac{H_{j-1}N}{N}$, $a = [h_i]$ $\quad b \in \dfrac{H_jN}{N} = [h_i]$

$b^{-1}ab = [\underbrace{h_b^{-1} h_a h_b}_{\in H_{j-1}}] = [\text{in } h_{j-1}]$

$h_b \in H_j$, $h_a \in H_{j-1}$

$$\frac{H_j \cdot N/N}{H_{j-1} N/N} \underset{\text{2nd Iso}}{=} \underbrace{\frac{H_j N}{H_{j-1} N}}_{= B} \xleftarrow{\phi} \frac{H_j}{H_{j-1}} = A$$

ϕ is onto

$B = A/\ker\phi$, so B is abelian

$\phi([h]_A) = [h]_B$

$h \in H_j$ claim 1. ϕ is well-defined

2. ϕ is onto.