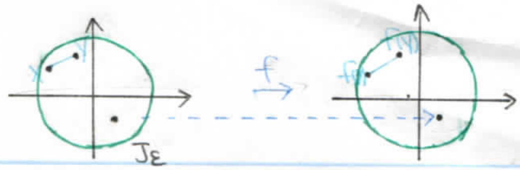


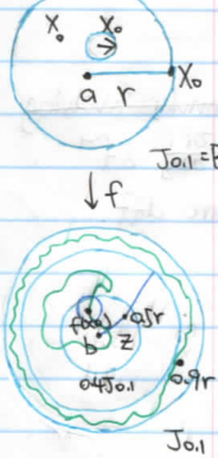
Oct 24

*Recall IFT & "Jelly Rigid"

*Part I: f is 1-1 on $J_{0.1}$

Indeed, if $f(x)=f(y)$, then $\|-(y-x)\| \le 0.1\|y-x\|$. $\Rightarrow$ $\|y-x\|=0 \Rightarrow y=x$.

Part II. f is onto $0.4J_{0.1}$



pf: Let $J_{0.1}=B(a,r)$. I'll show that every $z\in B(b,0.4r)$ is in image(f).

Assume $z \notin$ image of f, consider $d: \overline{J_{0.1}} \to \mathbb{R}_{\ge 0}$ $d(x)=\|f(x)-z\|$. d is cont. on ~~ar~~ compact set.

hence it attains its min at some $x_0\in\overline{J_{0.1}}$ as $z\notin \mathrm{im}(f)$, $d(x_0)>0$.

option 1. $x_0\in \mathrm{int}\,\overline{J_{0.1}}$, consider $x_1=x_0+\delta(z-f(x_0))$ ~~s.t. ① $\delta<1$ ② δ is sm~~

s.t. i) $\delta<1$ ii) δ is small enough s.t. $x_1\in J_{0.1}$.

Ex: show that $d(x_1)<d(x_0)$ then $\Rightarrow\Leftarrow$

option 2: $x_0\in \mathrm{bd}\,J_{0.1}$, $d(x_0)\ge 0.5r$, yet $d(b)\le 0.4r$, but then details left. $\Rightarrow\Leftarrow$

*. Take $\underset{b}{\underset{\Psi}{V}} = 0.4J_{0.1}$, $\underset{a}{\underset{\Psi}{U}} = f^{-1}(V)\subset J_{0.1}$ and by parts I & II, $f|_U$ is 1-1 and onto

Part III. $(f|_U)^{-1}$ is cont. $\|u-v\|\le\varepsilon\|v\| = \varepsilon\|u+(v-u)\| \le \varepsilon\|u\|+\varepsilon\|v-u\|$

$(1-\varepsilon)\|u-v\| \le \varepsilon\|u\|$.

$\|u-v\| \le \frac{\varepsilon}{1-\varepsilon}\|u\| \le 0.5\|u\|$ $\quad \|v\|-\|u\| \le \|u-v\| \le 0.5\|u\|$. $\quad \|v\|\le 1.5\|u\|$

$\|y-x\| \le 1.5\|f(y)-f(x)\|$ at $x=f^{-1}(z_1)$ & $y=f^{-1}(z_2)$. ~~this is $\|f^{-1}(z_1)$~~

this is $\|f^{-1}(z_1)-f^{-1}(z_2)\| \le 1.5\|z_1-z_2\|$ $\Rightarrow$ cont.

Oct 26.

*Term test: Tue Nov 1 5-7 pm. @ Banting BI 131.

Extra office Hour: Jeff Mon 4-7 pm @ Huron 215 10th floor ; Dror Tue 11-2 BA 6178

Material: everything to Fri (Oct 28) Roughly choose 4 are of 5

$\frac{1}{3}$ "prove as in class" $\frac{1}{3}$ solve "as in hw" $\frac{1}{3}$ "Fresh solve"

WLOG $Df(a)=I$. $a=b=0$. Done $V=0.4J_{0.1}$ $U=f^{-1}(V)$ $(f|_U)^{-1}$ exists & is cont.

Part IV. f^{-1} is diffable at $a=0$

proof: $f^{-1}(0+x)=f^{-1}(0)+(Df^{-1})(0)\,x+\text{small}$ $\quad f^{-1}(x)=0+I\cdot x+\text{small}$.

In TL, take $y=0$, get $|-f(x)+x| \le \varepsilon|-f(x)|$ on J_ε

meaning $|f(x)-x|\le\varepsilon|f(x)|$ on J_ε ← ignore this step.

tak $y=f(x)$ so $x=f^{-1}(y)$ on a suff. small nbd of 0. $|-y+f^{-1}(y)|\le\varepsilon|y|$.

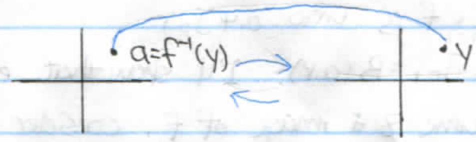
So $\frac{\|f^{-1}(y)-Iy\|}{\|y\|}<\varepsilon$ near 0.

So as $y\to 0$ get $\frac{\|f^{-1}(y)-f^{-1}(0)-Iy\|}{\|y\|}\to 0$

Part V: f^{-1} is diffable near a

repeat argument with other initial point.

Part VI: f^{-1} is C^r near a



pf: $(Df^{-1}(y)) = (Df(f^{-1}(y)))^{-1}$

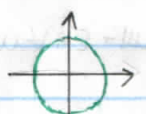
If f is C^2 then f^{-1} is C^1 $\quad (Df^{-1})(y) = Df(f^{-1}(y))^{-1}$ ~~= explicit formulas involving~~

$= $ explicit formulas involving $\frac{\partial F_i}{\partial x_g}$ & f^{-1}

Can differentiate RHS explictly, find that all entries of $Df^{-1}(y)$ are diff.

So f^{-1} is C^2 □.

Thm: The Implict function thm.



$y = g(x) = \pm\sqrt{1-x^2}$ or none if $x > 1$.

more precisely near a sol'n, under good condition, there are more sol^n.

$x^2+y^2-1 = f(x,y) = 0$, $x^2+y^2=1 \Rightarrow \exists g(x)$ s.t. $f(x, g(x)) = 0$.

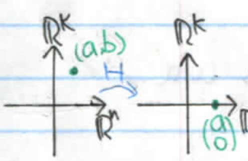
Thm (Imp FT) Given a C^r function $f: \mathbb{R}^n_{x_1 \dots x_n} \times \mathbb{R}^k_{y_1 \dots y_k} \to \mathbb{R}^k$ and $(a,b) \in \mathbb{R}^n \times \mathbb{R}^k$ s.t. $f(a,b) = 0$.
& $\frac{\partial f}{\partial y}$ is nonsingular. Then there exist a unique $g: \binom{\text{nbd } U}{\text{of } a} \to \binom{\text{nbd } V}{\text{of } b}$ s.t. $g(a) = b$
and $\forall x \in U$, $f(x, g(x)) = 0$, also $Dg = \cdots$

Oct 28.

proof $F(z,y) = 0 \iff \begin{cases} x = z \\ f(x,y) = 0 \end{cases}$ unknown x, y.

brilliant so now with $H\binom{x}{y} = \binom{x}{f(x,y)}$ $\quad H: \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ $\quad \iff H\binom{x}{y} = \binom{z}{0}$

then solvable using inv. function thm.



$H\binom{a}{b} = \binom{a}{F(a,b)} = \binom{a}{0}$ $\quad H$ is C^r

$DH\binom{a}{b}$ $\quad DH = \begin{pmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_1}{\partial y} \\ \frac{\partial H_2}{\partial x} & \frac{\partial H_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$ invertible?

$\frac{\partial f}{\partial y} = \begin{pmatrix} \partial f_1/\partial y_1 & \cdots & \partial f_1/\partial y_k \\ \vdots & & \vdots \\ \partial f_k/\partial y_1 & \cdots & \partial f_k/\partial y_k \end{pmatrix}$ $\quad \frac{\partial f}{\partial x} = \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & & \vdots \\ \partial f_k/\partial x_1 & \cdots & \partial f_k/\partial x_n \end{pmatrix}$

$\frac{\partial x}{\partial x} = \begin{pmatrix} \partial x_1/\partial x_1 & \cdots & \partial x_1/\partial x_n \\ \vdots & & \vdots \\ \partial x_n/\partial x_1 & \cdots & \partial x_n/\partial x_n \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ $\quad \frac{\partial x}{\partial y} = 0$.

DH is invertible $\iff \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial f}{\partial y} \end{pmatrix}$ is $\iff \frac{\partial f}{\partial y}$ is invertible, i.e. $\boxed{\frac{\partial f}{\partial y} \text{ is nonsingular}}$.

$\Rightarrow \exists H^{-1}$ in some nbd of (a,b) s.t. $g(z) = \pi_2\left(H^{-1}\binom{z}{0}\right)$

$\pi_2\ \mathbb{R}^{n+k} \to \mathbb{R}^k \quad (x,y) \mapsto y$.

easy to verify $g(a) = b$ & $\forall x$ $f(x, g(x)) = 0$. & g is C^r.

$0 = f(x, g(x))$

$\downarrow$ taking D

$0 = D[\ \ldots\ ]$

$= \left(\frac{\partial f}{\partial x}\ \ \frac{\partial f}{\partial y}\right)\begin{pmatrix} I \\ Dg \end{pmatrix}$

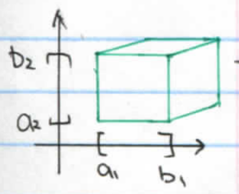
$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} Dg$

$$\mathbb{R}^n_x \xrightarrow[\begin{pmatrix}\partial x/\partial x \\ \partial g/\partial x\end{pmatrix}]{x \mapsto \begin{pmatrix}x \\ g(x)\end{pmatrix}} \mathbb{R}^n \times \mathbb{R}^k \xrightarrow[\left(\frac{\partial f}{\partial x}\ \ \frac{\partial f}{\partial y}\right)]{\begin{pmatrix}x \\ y\end{pmatrix} \mapsto f(x,y)} \mathbb{R}^k$$

$$\Rightarrow Dg = -\underbrace{\left(\frac{\partial f}{\partial y}\right)^{-1}}_{k\times k}\underbrace{\left(\frac{\partial f}{\partial x}(x, g(x))\right)}_{k\times n}$$

$k \times n$

$\square$



Aim of courses: stoke's thm $\int_M d\omega = \int_{\partial M} \omega$.

Test: Given $f: \mathbb{R}^n \to \mathbb{R}$ & $Q = \prod_{j=1}^{n} [a_j, b_j]$ Define $\int_Q f$.

TBC Next week.