Lecture 2 Extra Problem

Solution

September 16, 2016

QUESTION. Can all of $\mathbb{R}^3$ be covered by a set of disjoint non-degenerate circles? (Recall that a circle does not include its interior).

SOLUTION. Yes. To see why, we first must show that any sphere (the boundary points of a ball) missing 2 distinct points can be represented as a union of disjoint circles in $\mathbb{R}^3$.

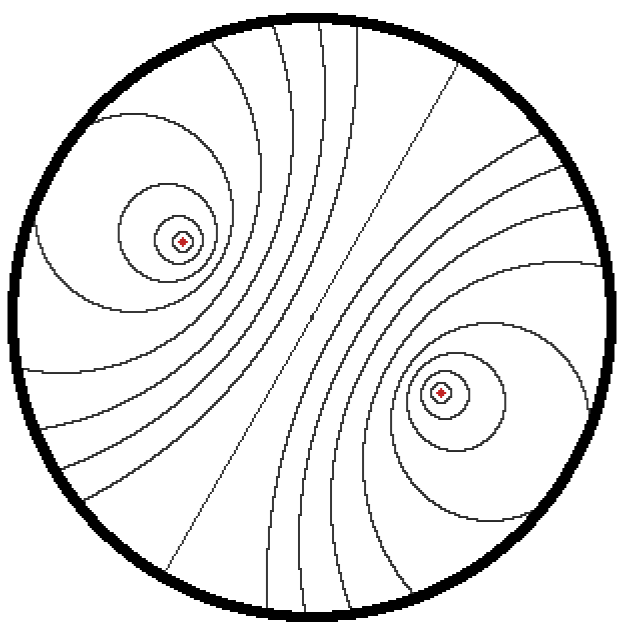

Figure 1: A sphere missing 2 distinct points (represented as red dots) and (some of) the disjoint circles that cover it.

Let S be a sphere missing 2 distinct points x and y. Let P_x, P_y be the planes tangent to the sphere at x and y, respectively. If the points are antipodal, then the tangent planes don't intersect. In this case, the set of all intersections between P_x and S as P_x moves towards P_y should do. If the points are not antipodal, the tangent planes intersect at some line L. In this case, the set of all intersections between P_x and S as P_x "swings about" L towards P_y does the trick (Figure 1).

Now consider the set of all spheres centred at the origin. If we can "puncture" each of these spheres in exactly 2 distinct places using disjoint circles, and also have one of these circles pass through the origin, we're done. The set

$$C = \{x = (x_0, x_1, x_2) \in \mathbb{R}^n |\, \exists z \in \mathbb{Z}.\, (x_0 - (1 + 4z))^2 + x_1^2 = 1 \wedge x_2 = 0\}$$

contains the origin, and intersects each sphere centred at the origin at exactly 2 distinct points. For each $r \in \mathbb{R}$, let S_r be the sphere of radius r centred at the origin. Let G_r be a set of disjoint circles covering $S_r \setminus C$. The set $(\bigcup_{r\in\mathbb{R}} G_r) \cup C$ contains disjoint circles covering all of $\mathbb{R}^3$. $\square$

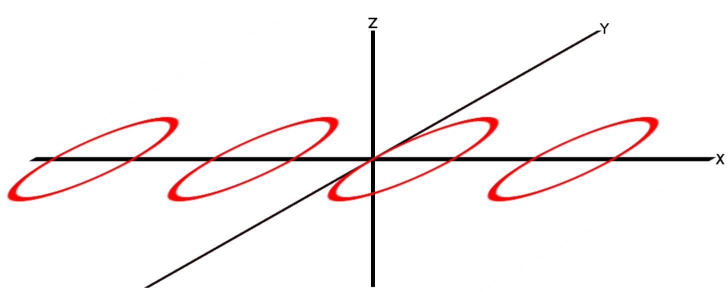

Figure 2: The set C.

As a final remark, note that the fact that $\mathbb{R}^n$ can be covered for $n > 3$ follows trivially from the fact that the set constructed above can be "tiled" along the 4^{th} dimension, after which we can continue tiling as necessary.