MAT 240 Nov. 21, 2006.

$Ax = b$
$EAx = Eb$
$A'x = b'$ A' is r.r.e.f.

<u>Determinants.</u>

$\det : M_{n\times n}(F) \to F$ $A \in M_{n\times n}(F)$, $\det(A) = |A| \in F$.

1. Usefulness
2. Formula
3. "Axiomatic" properties

1. Usefulness.

<u>Thm.</u> A is invertible iff $\det A \neq 0$ i.e. iff $\det A$ is invertible

2. Formula.

$\det$, $|\cdot|$ is defined recursively for $n\times n$ matrices in terms of $(n-1)\times(n-1)$ matrices.

1. $\det(a_{11}) = a_{11}$
2. If $A = (A_{ij})$ is $n\times n$, define
$$\det A = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1j})$$

$\tilde{A}_{ij}$:= the matrix get from A removing i^{th} j^{th} column.

<u>eg.</u> $$\overbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}^{A} \to 1\cdot\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\cdot\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\cdot\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

<u>Ex.</u> $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ $(= a\cdot\det(d) - b\cdot\det(c))$

$$= 1\cdot(5\cdot 9 - 6\cdot 8) - 2(4\cdot 9 - 6\cdot 7) + 3(4\cdot 8 - 5\cdot 7)$$
$$= -3 - 2(-6) + 3(-3)$$
$$= -3 + 12 - 9$$
$$= 0.$$

$\Rightarrow A$ is not invertible. (2^{nd} row is avg. of 1^{st} & 3^{rd} rows ∴ it's lin. comb. of them ∴ 3 rows not lin. in

Properties.

a) $\det(I) = 1$

b) $\det$ is multi-linear in the rows.

$$\det\left(\underbrace{\begin{bmatrix} - r_1 - \\ - r_2 - \\ r_j \\ - r_n - \end{bmatrix}}_{n}\right), \quad r_j = a r_j' + b r_j'' \quad = a \det \begin{bmatrix} r_1 \\ r_j' \\ r_n \end{bmatrix} + b \det \begin{bmatrix} r_1 \\ r_j'' \\ r_n \end{bmatrix}.$$

Aside: $x \cdot y$ is "multi-linear" in x & y.

$(5+3)y = 5y + 3y$

$x(7+9) = x \cdot 7 + x \cdot 9$

But,

$(5+3)\cdot(7+9) \neq 5 \cdot 7 + 3 \cdot 9$.

c) If ~~[illegible]~~ a pair of adjacent rows in A are equal, $\det A = 0$.

$$\det\left(\begin{bmatrix} - r_1 - \\ - r_1 - \\ * \end{bmatrix}\right) = 0. \quad \text{or} \quad \det\left(\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}\right)$$

Thm. It is easy to tell how dets change under row ops.

1. Interchanging 2 rows.
   $\det(E^1_{ij} A) = -\det A \quad \Rightarrow \det(E^1_{ij}) = -1.$

2. Multiplying a row by c.
   $\det(E^2_{i,c} A) = c \cdot \det A$, even for $c = 0$.
   $\Rightarrow \det(E^2_{i,c}) = c$.

3. $\det(E^3_{i,j,c} A) = \det A \quad \Rightarrow \det(E^3_{i,j,c}) = 1.$

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eg

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{vmatrix} =$$

$$= -(-2)\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2 \cdot 1 = 2$$

Moral: Using elementary row ops, we can reduce the computation of any det to the computation of the det of a matrix in r.r.e.f.

If A' is in r.r.e.f

$$\det A' = \begin{cases} 0 & \text{if } A' \text{ has a row of } 0\text{'s} \\ 1 & \text{otherwise, } A' = I \end{cases}$$

Moral: Using the properties, we can compute any det w/o going back to formulas.

Moral: $\det A \neq 0$ iff A is invertible.

Pf Do row reduction without ever multiplying a row by 0.
Find that $\det A = \underbrace{\gamma}_{\neq 0} \cdot \det(\underbrace{EA}_{\text{is in r.r.e.f}})$

$\neq 0$. γ is a product of c's & -1's $\Rightarrow \gamma \neq 0$.

tible $\overset{\Leftrightarrow}{\text{iff}}$ $EA = I$ $\overset{\Leftrightarrow}{\text{iff}}$ $\det(EA) \neq 0$ $\overset{\Leftrightarrow}{\text{iff}}$ $\det A \neq 0$.

follows from properties

Proof of Thm from "Axiomatic" properties:

(2) follows from b taking $r''=0$ or $b=0$

$$\begin{vmatrix} r_1 \\ r_2 \\ \vdots \\ r_j + 7 r_{j+1} \\ r_{j+1} \\ \vdots \\ r_n \end{vmatrix} \overset{(b)}{=} \begin{vmatrix} r_1 \\ \vdots \\ r_j \\ r_{j+1} \\ \vdots \\ r_n \end{vmatrix} + 7 \begin{vmatrix} r_1 \\ \vdots \\ r_{j+1} \\ r_{j+1} \\ \vdots \\ r_n \end{vmatrix} \overset{(c)}{=} \begin{vmatrix} r_1 \\ \vdots \\ \vdots \\ r_n \end{vmatrix} + 0$$

$\Rightarrow$ (3) holds if the two rows are adjacent.

$$\begin{vmatrix} \hline r' \\ r'' \\ \hline \end{vmatrix} \overset{\text{by}}{=} \begin{vmatrix} \hline r' \\ r''+r' \\ \hline \end{vmatrix} \overset{1-2}{=} \begin{vmatrix} \hline -r'' \\ r''+r' \\ \hline \end{vmatrix} \overset{2+1}{=} \begin{vmatrix} \hline -r'' \\ r' \\ \hline \end{vmatrix} \underset{\text{by (2)}}{\overset{-1}{=}} - \begin{vmatrix} \hline r'' \\ r' \\ \hline \end{vmatrix}$$

$\Rightarrow$ (1) holds for adjacent rows

$\Rightarrow$ (3) holds even if the relevant rows (r_i & r_j) are not adjacent.

b/c:

$$m\{ \begin{vmatrix} r_i \\ \hline r_j \\ \hline \end{vmatrix} = \pm^{m\{} \begin{vmatrix} \hline r_i \\ r_j \\ \hline \end{vmatrix} \overset{m\{}{=\pm} \begin{vmatrix} \hline r_i \\ r_j + 7 r_i \\ \hline \end{vmatrix} = m\{ \begin{vmatrix} r_i \\ \hline r_j + 7 r_i \\ \hline \end{vmatrix}$$

no sign b/c
now flip back
∴ even # of ops
so (+) sign.

$\Rightarrow$ (1) holds for any pair of rows

Pf α: the computer trick again.

Pf β

b/c

$$m\{ \begin{vmatrix} \hline r_i \\ \hline r_j \\ \hline \end{vmatrix} \xrightarrow[\text{switches}]{2m+1} (-1)^{2m+1} \begin{vmatrix} r_j \\ \hline r_i \\ \hline \end{vmatrix} = -1 \begin{vmatrix} r_j \\ \hline r_i \end{vmatrix}$$

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$$\det\left(I_{n\times n}\right) = \det\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ & & \ddots & \end{pmatrix} = 1\cdot\det I_{n-1} - 0\cdot\det(\ ) + 0\cdot\det(\,) + $$

$$= \det I_{n-1}$$

$$= \det I_{n-2}$$

$$= \cdots$$

$$= \det 1$$

$$= 1.$$

$\square$.