example in F_2

$27 \rightsquigarrow \tau(27) = \tau(26+1) = \tau(26) + 1_{F_2}$

$= \tau(2 \cdot 13) + 1 = \tau(2) \cdot \tau(13) + 1$

$= (1_F + 1_F)\,\tau(13) + 1$

$0 = 0 \cdot \tau(13) + 1 = 1$.

Def. The "characteristic" of a field F is the least integer (positive) n s.t. $\tau(n) = 0$, if such an integer exits. Otherwise, it is 0. It is divided char(F)

Eg. char($\mathbb{R}$) τ maps $\mathbb{Z}$ to itself $\to \mathbb{R}$.

$\Rightarrow$ Char($\mathbb{R}$) = 0

Char($\mathbb{Q}$) = 0.

Char(F_2) = 2 $\quad \tau(1) = 1 \neq 0 \quad \tau(2) = 1_F + 1_F = 0_F$

$\tau(0) = 0$

char(F_7) = 7 $\qquad F_7 = \{0, 1, 2, \cdots 6\}$

claim char(F) is either 0 or a prime number

PF: suppose by contradiction

$\mathrm{Char}(F) = m \cdot n$ where $m, n > 1$, $m, n < \mathrm{char}(F)$

Then $0 = \tau(m \cdot n) = \tau(m) \cdot \tau(n)$

So either $\tau(m) = 0$ or $\tau(n) = 0$.

if $\tau(m) = 0$, then we found a positive integer small than char(F) s.t. $\tau(\text{that integer}) = 0$

contradicts the minimullity of Char F

likewise if $\tau(n) = 0$.

In reality, the notation "τ" is almost never used.

In practice, people write 7 for $\tau(7)$ or maybe 7_F

The complex number

$+, \cdot, -, \div, \sqrt{\ }$

Thm. There is a field $\mathbb{C}$ that contains $\mathbb{R}$, and in addition has an element i, s.t $i^2 = -1$

dream, implications, formalization, proof

Dream: "i exists" "$\mathbb{C}$ exists"

$\mathbb{C} = \{7, i, 0, 1, 7 \cdot i, 3, 3+7i, \forall a,b \in \mathbb{R}\ a+bi\}$

$$(a+bi)(c+di) = a(c+di) + bi(c+di)$$
$$= ac + adi + bic + bidi$$
$$= (ac + bdi^2) + (ad+bc)i$$
$$= \underbrace{(ac-bd)}_{\text{"real part"} \in \mathbb{R}} + \underbrace{(ad+bc)}_{\in \mathbb{R}.\ \text{"imaginary part"}} i$$

$$(a+bi)^{-1} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}$$
$$= \underbrace{\frac{a}{a^2+b^2}}_{\in \mathbb{R}} + \underbrace{\frac{-b}{a^2+b^2}}_{\in \mathbb{R}} \cdot i$$

Definition

$\mathbb{C} = \{(a,b) : a,b \in \mathbb{R}\}$.

Turn this into a Field candidate

by $0_{\mathbb{C}} = (0,0)$ $1_{\mathbb{C}} = (1,0)$

$(a,b) + (c,d) = (a+c, b+d)$.