MAT257 Midterm 3 Review

2017-03-11

Theorem. Given $\underset{\subset \mathbb{R}^n}{A} \xrightarrow[\text{onto}]{g \; 1\text{-}1} \underset{\subset \mathbb{R}^n}{B} \xrightarrow{f} \mathbb{R}$, $\int_B f = \int_A (f \circ g) |\det Dg|$

Corollary. Let $P(v_1, \dots, v_n)$ be the "parallelepiped" spanned by $v_1, \dots, v_n$:

$P(v_1, \dots, v_n) = \{ \sum a_i v_i : 0 \le a_i \le 1 \}$. Then:

$\mathrm{Vol}(P(v_1, \dots, v_n)) = |\det(v_1 | v_2 | \dots | v_n)|$.

Definition. $h: \mathbb{R}^n \to \mathbb{R}^n$ is called an "isometry" if $\forall x, y \; d(h(x), h(y)) = d(x, y)$.

Theorem. $h: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry IFF it can be written in the form $h(x) = p + Ax$ where $p \in \mathbb{R}^n$, $A \in M_{n \times n}$ such that $A^T A = I$. (vol. preserving)

Definition. $O(n) = \{ A \in M_{n \times n}(\mathbb{R}) : A^T A = I \}$ orthogonal/rotational matrices:

1. $A, B \in O(n) \Rightarrow AB \in O(n)$
2. $\exists I \in O(n)$ such that $AI = IA = A \quad \forall A \in O(n)$.
3. $\forall A \in O(n) \; \exists B \in O(n)$ such that $AB = BA = I$.

Proof. ($\Leftarrow$) Given $h(x) = p + Ax$, show $d(h(x), h(y)) = d(x, y)$.

($\Rightarrow$) 1. Show WLOG $h(0) = 0$.

2. h preserves norms
3. h preserves inner products.
4. $A = (h(e_1) | \dots | h(e_n)) \in O(n)$
5. h is linear

Gram-Schmidt

$v_1' = u_1$ $\qquad v_1 = \pm v_1' / \|v_1'\|$

$v_2' = u_2 - \langle u_1, v_1 \rangle v_1$ $\qquad v_2 = \pm v_2' / \|v_2'\|$

$\vdots$

$v_k' = u_k - \sum_{j=1}^{k-1} \langle u_k, v_j \rangle v_j$ $\qquad v_k = \pm v_3' / \|v_3'\|$

Theorem. There is a unique $V:(\mathbb{R}^n)^k \to \mathbb{R}_{\geq 0}$ such that:

① If $h:\mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation & $x_i \in \mathbb{R}$,
$V(h(x_1),\dots,h(x_k)) = V(x_1,\dots,x_k)$.

② If $x_i \in \mathbb{R}^k \times \{0\}$, so $x_i = \binom{y_i}{0}$ w/ $y_i \in \mathbb{R}^k$, then
$V(x_1,\dots,x_k) = |\det(y_1|\dots|y_k)|$.

③ $V(x_1,\dots,x_k) = 0 \iff \{x_i\}$ is linearly dependent.

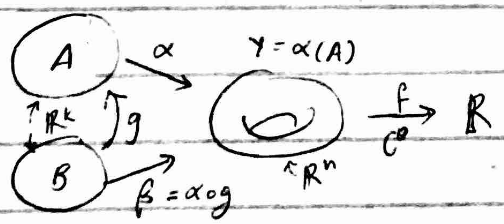
④ $X = (x_1|\dots|x_k) \in M_{n\times k} \Rightarrow V(x_1,\dots,x_k) = |\det(X^T \cdot X)|^{1/2}$.

Definition. A parametrized k-manifold in $\mathbb{R}^n$ is a C^1 map $\alpha: A \to \mathbb{R}^n$ where $A \subset \mathbb{R}^k$ is some open set. $\alpha(A) = Y$
↳ the manifold
↳ the parametrization

Definition. If $Y = \alpha(A)$, $V(Y) = \int_A V(D\alpha) = \int_A |\det(D\alpha)^T(D\alpha)|^{1/2}$.

Theorem. If $\alpha: A \to \mathbb{R}^n$ is a parametrized manifold and $g: B \to A$ is a diffeomorphism, then set $\beta = \alpha \circ g$: $\alpha(A) = \alpha(B)$, $V(Y,\alpha) = V(Y,\beta)$.
($B \subset \mathbb{R}^k$, $A \subset \mathbb{R}^k$)

Definition.

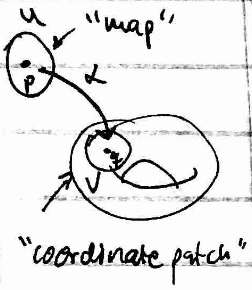


If $g: B \to A$ is a diffeomorphism of open sets in $\mathbb{R}^k$ and $\alpha: A \to \mathbb{R}^n$ is a manifold, set $\beta = \alpha \circ g$, then $\alpha(A) = \beta(B) = Y$ and $V(Y,\alpha) = V(Y,\beta)$.

Definition. $\int_Y f\,dV = \int_A (f \circ \alpha)\, V(D\alpha)$

Theorem. $\int_Y f\,dV = \int_A (f\circ\alpha)\, V(D\alpha) \overset{*}{=} \int_B (f \circ \beta)\, V(D\beta)$.

Stokes Theorem. $\int_M d\omega = \int_{\partial M} \omega$.



Definition. A C^r k-manifold w/o boundary in $\mathbb{R}^n$ is a subset $M \subseteq \mathbb{R}^n$ such that $p \in M$ has an open nbd $V \in M$ such that there is an open nbd $U \subseteq \mathbb{R}^k$ and a C^r homeomorphism (1-1, onto, cont.) $\alpha: U \to V$ whose differential has rank k $\forall x \in U$.

Note: Let $H^k = \{x \in \mathbb{R}^k : x_k \geq 0\}$ denote the k-dim upper half-space

Definition. A C^r k-manifold w/ boundary in $\mathbb{R}^n$ is a subset $M \subseteq \mathbb{R}^n$ such that $p \in M$ has an open nbd $V \in M$ such that there is an open nbd $U \subseteq H^k$ and a C^r homeomorphism (1-1, onto, cont.) $\alpha: U \to V$ whose differential has rank k $\forall x \in U$.
The boundary, ∂M, of M is:

$$\partial M = \left\{ p \in M : \begin{array}{l} \text{for some coordinate patch } \alpha, \ p = \alpha(q) \\ q \in \partial H^k := \mathbb{R}^{k-1} \times \{0\} \end{array} \right\}.$$

Theorem. ∂M is a $(k-1)$-manifold of class C^r w/o boundary.

Definition. Let $S \subseteq \mathbb{R}^k$ be arbitrary. A function $f: S \to \mathbb{R}^n$ is said to be of class C^r if $\forall p \in S$, there is an open subset U_p of $\mathbb{R}^k$ containing p and a C^r function $g_p: U_p \to \mathbb{R}^n$ such that $g_p(x) = f(x)$ $\forall x \in S \cap U_p$.

Theorem. Let $S \subseteq \mathbb{R}^k$ be arbitrary. A function $f: S \to \mathbb{R}^n$ is of class C^r IFF there exists an open subset U of $\mathbb{R}^k$ containing p and a C^r function $g: U \to \mathbb{R}^n$ such that $g(x) = f(x)$ $\forall x \in S$.

Proposition. If M^k is a class C^r manifold in $\mathbb{R}^n$ and $\alpha: \underset{\subseteq \mathbb{R}^k}{U} \to \underset{\subseteq M^k}{V}$ is a coordinate patch (homeo, diffable, maxrank differential), then $\alpha^{-1}: \underset{\subseteq \mathbb{R}^n}{V} \to U$ is also C^r.

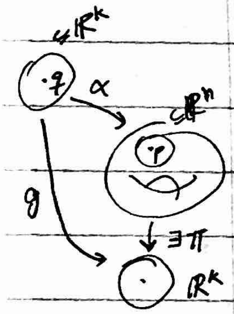
Lemma. Partitions of Unity

Given a collection $\mathcal{A}$ of open sets in $\mathbb{R}^k$ whose overall union is $A = \bigcup_{U \in \mathcal{A}} U$, there exists a sequence $\{\phi_i\}$ of non-negative compactly supported C^∞ functions such that:

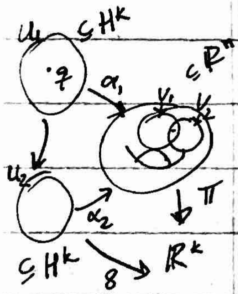
① For all i, $\exists U \in \mathcal{A}$ s.t. $\operatorname{supp}(\phi_i) \subseteq U$.

② For all $x \in A$, $\exists$ nbhd V s.t. $\{ i : V \cap \operatorname{supp}(\phi_i) \neq \emptyset \}$ is finite.

③ $\sum_{i=1}^{\infty} \phi_i = 1$ on A.



Theorem. Given a coordinate patch $\alpha : U \to M$, with $U \subseteq H^k$ and $M \subseteq \mathbb{R}^n$, and any $p = \alpha(q)$, there exists a projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ (defined by ignoring the $(n-k)$ coordinates of $\mathbb{R}^n$), such that $g = \pi \circ \alpha$, then g is a diffeomorphism in the vicinity of q.



Corollary. Transition functions σ are C^r.

$\sigma = \alpha_2^{-1} \circ \alpha_1$, on $\alpha_1^{-1}(V_1 \cap V_2)$.

$(\alpha_2^{-1} = g^{-1} \circ \pi)$.

Theorem. If $A \subseteq \mathbb{R}^n$ is open and $f : A \to \mathbb{R}$ is C^r, then if $h \in \mathbb{R}$ is a height s.t. $p \in f^{-1}(h)$, $df(p)$ is of rank 1 ($\Leftrightarrow df(p) \neq 0$), then $N = f^{-1}(h)$ is a manifold and so is $M = f^{-1}((-\infty, h])$. Finally, $\partial M = N$.