~~IVT Proof~~

Def'n: X is "connected" if it has no separation (i.e. has no non-trivial clopen sets).

IVT Proof: Assume not, no x s.t. $f(x)=0$. Then, write $X = f^{-1}(-\infty,0) \cup f^{-1}(0,\infty) = U_0 \cup U_1$
By assumption, $x_0 \in U_0$, $x_1 \in U_1$ $\Rightarrow$ non-empty.
So, have a separation of X, $\Rightarrow$ X not connected, a contradiction.

Theorem: $I=[0,1]$ is connected.

Proof: Assume $A \subset [0,1]$ is clopen. Then, A^c clopen.
WLOG, say $0 \in A$. Want to show here that $A = I$.
Let $G = \{g : [0,g] \subset A\}$.
Let $g_0 = \sup G$.
Claim: G non empty ($\sup G$ exists).
Proof: $0 \in A$, A open $\Rightarrow$ some nbhd of 0 is in A
$\Rightarrow$ $[0,\varepsilon) \subset A$ for some $\varepsilon > 0$
so $[0, \frac{\varepsilon}{2}] \subset A$ $\Rightarrow$ $\frac{\varepsilon}{2} \in G$.

So $\sup G$ makes sense.

By above claim, $g_0 > 0$.
Assume $g_0 < 1$, $g_0 = \sup G$.
By def'n of G, $\forall \varepsilon > 0$ $\exists g_\varepsilon > g_0 - \varepsilon$ s.t. $g_\varepsilon \in G$. By property of sup.
But then, $g_\varepsilon \in A$ (since in G).
So since A closed, $g_0 \in A$.
Since A open, for some $\delta > 0$, $(g_0 - \delta, g_0 + \delta) \subset A$
So, $\underbrace{[0, g_\delta] \cup (g_0 - \delta, g_0 + \delta)}_{= [0, g_0 + \delta)} \subset A$, $\Rightarrow$ $[0, g_0 + \frac{\delta}{2}] \subset A$
$\Rightarrow$ $g_0 + \frac{\delta}{2} \in G$, contradicting $g_0 = \sup G$
$\Rightarrow$ $g_0 = 1$.