

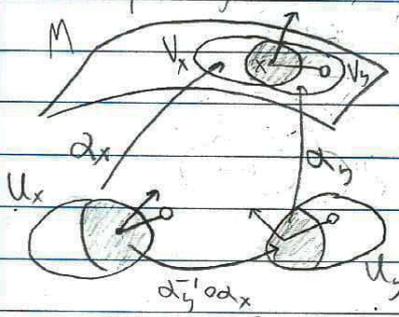
\*Let  $M$  be a  $k > 1$ -manifold in  $\mathbb{R}^n$

- $M$  is Munkus-orientable if  $\exists \{\alpha_x: U_x \rightarrow V_x\}_{x \in M}$  where  $\alpha_x$  is a coordinate patch onto  $x$  in  $M \forall x \in M$  &  $\forall x, y \in M, \alpha_y^{-1} \circ \alpha_x$  is positive where it exists, i.e.  $\det[D(\alpha_y^{-1} \circ \alpha_x)] > 0$  where it exists
- $M$  is D-orientable if  $\exists \{\mathcal{O}_x\}_{x \in M}$  where  $\mathcal{O}_x$  is an orientation of  $T_x(M) \forall x \in M$  &  $\forall x \in M \exists$  nbh of  $x$  in  $M, W_x \mid \exists F_1, \dots, F_k \in C^\infty$  vector fields on  $W_x$  s.t.  $\forall w \in W_x, (F_1(w), \dots, F_k(w)) \in \mathcal{O}_w$

Obviously, these definitions are very different. Dvor's is intuitive but Munkus is used for Stokes Thm.

Thm:  $M$  is M-orientable  $\iff M$  is D-orientable

pf: easy  $\Rightarrow \forall x \in M$ , let  $\mathcal{O}_x$  be the class of  $\alpha_x((\alpha_x^{-1}(x), e_1), \dots, (\alpha_x^{-1}(x), e_k)) = ((x, \frac{\partial \alpha_x}{\partial x_1}), \dots, (x, \frac{\partial \alpha_x}{\partial x_k}))$



Now that we've chosen our orientations, we have to show other patches obey it.

$$\begin{aligned} \text{Let } z \in M \mid x \in U_x, z \in U_y, \alpha_y((\alpha_y^{-1}(z), e_1), \dots, (\alpha_y^{-1}(z), e_k)) &= (d\alpha_x \circ d\alpha_y^{-1}) \circ \alpha_y((\alpha_y^{-1}(z), e_1), \dots, (\alpha_y^{-1}(z), e_k)) \\ &= (d\alpha_x \circ d\alpha_y^{-1}) \circ \alpha_y((\alpha_x^{-1}(z), Z e_1), \dots, (\alpha_x^{-1}(z), Z e_k)) \\ &= \alpha_x((\alpha_x^{-1}(z), Z e_1), \dots, (\alpha_x^{-1}(z), Z e_k)) \mid \text{where } Z = D(\alpha_x^{-1} \circ \alpha_y)(\alpha_y^{-1}(z)), \text{ so } \det Z > 0 \end{aligned}$$

$Z$  can be thought to be a change of basis into its image.

$$\{e_i\}_{i=1}^k \sim Z e_i \Rightarrow \{(\alpha_x^{-1}(z), e_i)\}_{i=1}^k \sim \{(\alpha_x^{-1}(z), Z e_i)\}_{i=1}^k$$

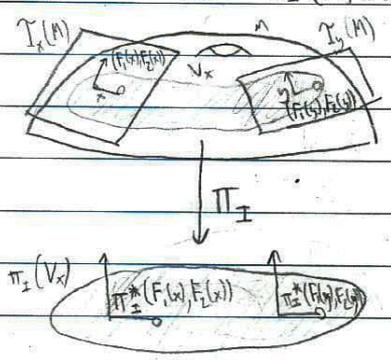
As  $d\alpha_x: T_{\alpha_x^{-1}(z)}(\mathbb{R}^k) \rightarrow T_x(M)$  is injective,  $\alpha_x((\alpha_x^{-1}(z), e_1), \dots, (\alpha_x^{-1}(z), Z e_k)) \in \mathcal{O}_x$

All that's left is to define the vector fields & show they hold the property

$\forall x \in M \forall i \in k$  let  $F_i$  be a vector field on  $U_x \mid \forall y \in U_x, F_i \mapsto d\alpha_x((\alpha_x^{-1}(y), e_i))$

$$\forall w \in U_x, (F_i(w))_{i=1}^k = (d\alpha_x((\alpha_x^{-1}(w), e_1), \dots, (\alpha_x^{-1}(w), e_k)))_{i=1}^k \in \mathcal{O}_w \text{ as we've just shown}$$

hard  $\Leftarrow \mathcal{I}(M)$  is very difficult to work on so this proof will work in projections of  $M$ . Our 1st task is



to utilize the vector fields to show  $M$ 's "nice" connected projections have constant orientation.

Let  $x \in M, \alpha_x$  its coord. patch. Restrict  $\alpha_x$ 's domain so it's connected (thus its range is  $\mathbb{R}^k$ )

& restrict its range so it's  $\subseteq W_x$ . As well,  $D\alpha_x(x)$  has  $k$  independent rows in positions  $i \in \{1, \dots, k\}$

As  $D\alpha_x(x)$ 's entries are continuous,  $\pi_I(D\alpha_x(x))$  is continuous. Restrict  $\alpha_x$ 's domain to  $\pi_I^{-1}(\mathbb{R}^k)$

so those rows are independent over the domain. Finally,  $g_{x,z} = \pi_I \circ \alpha_x$  has been shown in class to be a diffeo. on a nbh of  $x$ , restrict  $\alpha_x$ 's domain so  $g_{x,z}$  is a diffeo over it.

Call  $\alpha_x: U_x \rightarrow V_x, \pi_I \circ \alpha_x = g_{x,z}$  is a diffeo. as  $g_{x,z}^{-1} \circ \alpha_x^{-1}$  are, thus,  $\pi_I(V_x)$  is connected.

$\forall i \in k$ , let  $F_i(w) = (x, f_i(w)) \forall w \in W_x$ , so we are concerned with only  $\{D\pi_I(w), f_i(w)\}_{i=1}^k$

$\forall i \in k, f_i$ , entries of  $D\pi_I$  are  $C^\infty$  so  $\det(D\pi_I f_i)_{i=1}^k$  is continuous on  $V_x$ . As  $V_x$  is connected, if  $\{D\pi_I f_i\}_{i=1}^k$  switches orientation,  $\exists z \in V_x \mid \det(D\pi_I f_i(z))_{i=1}^k = 0 \Rightarrow \{D\pi_I f_i(z)\}_{i=1}^k$  is dependent  $\Rightarrow \{F_i(z)\}_{i=1}^k$  is dependent.

But this is impossible as  $\pi_I: \mathcal{I}_z(M) \rightarrow \mathcal{I}_z(\mathbb{R}^k)$  is an isomorphism &  $\{F_i\}_{i=1}^k$  is a basis of  $\mathcal{I}_z(M)$