MAT257 Final Review

<u>Claim.</u> If $\phi: \mathbb{R}^n_{x_i} \to \mathbb{R}^n_{y_i}$ and $\omega = f\,dy_{\underline{n}} \in \Omega^{top}(\mathbb{R}^n_{y_i})$,
then $\phi^*(\omega) = \underbrace{\det(D\phi)}_{J_\phi} \cdot \phi^* f \cdot dx_{\underline{n}} \in \Omega^{top}(\mathbb{R}^n_{x_i})$.

<u>Ex.</u> $\phi^*(dx \wedge dy) = d(r\cos\theta) \wedge d(r\sin\theta)$.

$$= J_\phi(x) \cdot \phi^* f \cdot dx_n = r \cdot 1 \cdot dr \wedge d\theta.$$

$$= r\, dr \wedge d\theta.$$

<u>Proof.</u> $\phi^*(dy_{\underline{n}}) = \phi^*(dy_1 \wedge \dots \wedge dy_n) = d(\phi^* y_1) \wedge \dots \wedge d(\phi^* y_n)$

$$= d\phi_1 \wedge \dots \wedge d\phi_n$$

$$= \left(\frac{\partial \phi_1}{\partial x_1} \cdot dx_1 + \frac{\partial \phi_1}{\partial x_2} \cdot dx_2 + \dots + \frac{\partial \phi_1}{\partial x_n} \cdot dx_n\right) \wedge$$

$$\left(\frac{\partial \phi_2}{\partial x_1} \cdot dx_1 + \frac{\partial \phi_2}{\partial x_2} \cdot dx_2 + \dots + \frac{\partial \phi_2}{\partial x_n} \cdot dx_n\right) \wedge \dots \wedge$$

$$\left(\frac{\partial \phi_n}{\partial x_1} \cdot dx_1 + \frac{\partial \phi_n}{\partial x_2} \cdot dx_2 + \dots + \frac{\partial \phi_n}{\partial x_n} \cdot dx_n\right).$$

$= \sum_{\text{pick one term from each line}}$ wedge these n terms together.

$= \sum_{\text{pick one term from each row \& col.}}$ wedge these n terms together.

$$= \sum_{\sigma \in S_n} \frac{\partial \phi_1}{\partial x_{\sigma_1}} \cdot dx_{\sigma_1} \wedge \dots \wedge \frac{\partial \phi_n}{\partial x_{\sigma_n}} \cdot dx_{\sigma_n}$$

σ_i : sequence of terms in line i, $1 \le i \le n$.

$$= \left(\sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \frac{\partial \phi_i}{\partial x_{\sigma_i}}\right) \cdot dx_{\underline{n}} = \det\left(\frac{\partial \phi_i}{\partial x_j}\right) \cdot dx_{\underline{n}}$$

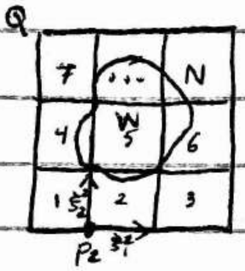
$$= \det(D\phi) \cdot dx_{\underline{n}}$$

If $\omega = \sum_{I \in \binom{n}{k}} a_I\, dy_I \in \Omega^k(\mathbb{R}^n_{y_i})$, then:

$(D\phi)_{J,I}$ $k \times k$ matrix
rows — J
cols — I

$$\phi^*(\omega) = \sum_{I \in \binom{n}{k}} \sum_{J \in \binom{n}{k}} \left(\phi^* a_I \cdot \det(D\phi)_{J,I}\right) dx_J$$

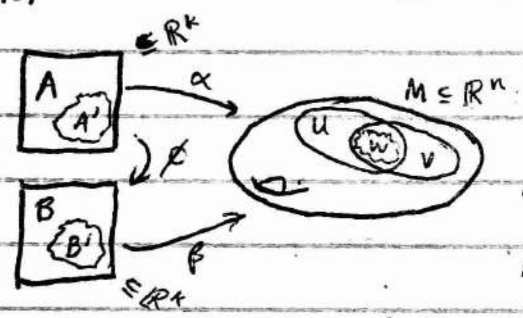
Suppose $\omega \in \Omega^{top}(\mathbb{R}^k_{x_i})$, $supp(\omega) = \left\{ \begin{matrix} \text{set of } x \in \mathbb{R}^k \\ \text{s.t. } \omega(x) \neq 0 \end{matrix} \right\} \subseteq Q$ (rectangle)



$$\int_Q \omega = \int_{\mathbb{R}^k} \omega := \int_{\mathbb{R}^k} f = \int_Q f \qquad \omega = f \cdot dx_{\underline{n}}$$

Intuitively, $\oint_Q \omega \underset{N\to\infty}{\sim} \sum_{i=1}^{N} \omega(\vec{\xi}_1^{\,i}, \vec{\xi}_2^{\,i}, \dots, \vec{\xi}_k^{\,i})$

$$= \sum_{i=1}^{N} f(p_i) \cdot (dx_1 \wedge \dots \wedge dx_k)(\vec{\xi}_1^{\,i}, \dots, \vec{\xi}_k^{\,i}) = \sum_{i=1}^{N} f(p_i) \cdot |\vec{\xi}_1^{\,i}| \cdot \ldots \cdot |\vec{\xi}_2^{\,i}|$$



$$= \sum_{i=1}^{N} f(p_i) \cdot V(R_i) \underset{N\to\infty}{\sim} \int_Q f.$$

$$\overset{\alpha}{\oint_M} \omega := \int_A \alpha^* \omega = \int_A f \quad \text{where } \alpha^*\omega = f dx_{\underline{k}}$$

Precisely, $\overset{\alpha}{\oint_M} \omega = \pm \overset{\beta}{\oint_M} \omega$ provided $supp\, \omega \subseteq V = im \beta$, and $\phi = \beta^{-1} \circ \alpha$ is C^∞. provided that $U \cap V$ is connected.

<u>Proof.</u> $\overset{\beta}{\oint_M} \omega = \int_B g$, where $\beta^* \omega = g\, dx_{\underline{k}}$.

$$\overset{\alpha}{\oint_M} \omega = \int_{A' = \alpha^{-1}(U \cap V)} \alpha^* \omega \overset{\alpha = \beta\circ\phi}{=\!=\!=} \int_{A'} (\beta \circ \phi)^* \omega = \int_{A'} \phi^*(\beta^* \omega)$$

$$= \int_{A'} \phi^*(g \cdot dx_{\underline{k}}) = \int_{A'} (g \circ \phi) \cdot \det(D\phi) \cdot dx_{\underline{k}}$$

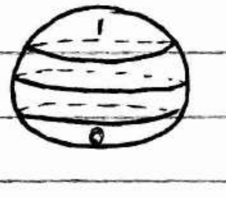
$$= \int_{A'} (g \circ \phi) \cdot \det(D\phi) = + \int_{A'} (g \circ \phi) \cdot |\det(D\phi)|.$$

on a connected set $\det(D\phi)$ is uniformly + or uniformly −.

$$= \pm \int_{B' = \beta^{-1}(U \cap V)} g = \pm \int_B g = \pm \overset{\beta}{\oint_M} \omega.$$

$\longrightarrow$

Find a PoU $\phi_i : M \to [0,1]$ smooth that is subordinate to "positive charts of M": * $\operatorname{supp} \phi_i \subseteq \operatorname{im} \alpha$, where α a positive chart.



* $\sum \phi_i = 1$
* local finiteness (finite 1's in any nbd).

<u>Definition.</u> $\overset{\phi}{\oint_M} = \sum_{i \in I} \int_M \phi_i \cdot \omega$.

<u>Proposition.</u> If ϕ_i & ψ_j are PoU, then $\overset{\phi}{\oint_M} \omega = \overset{\psi}{\oint_M} \omega = \int_M \omega$.

<u>Proof.</u> $\overset{\phi}{\oint_M} \omega = \sum_{i \in I} \int_M 1 \cdot \phi_i \cdot \omega = \sum_{i \in I} \int_M (\sum_j 1 \cdot \psi_j) \phi_i \cdot \omega$

$= \sum_i \sum_j \int_M \psi_j \cdot \phi_i \cdot \omega = \sum_j \sum_i \int_M \phi_i \cdot \psi_j \cdot \omega$

$= \sum_j \int_M (\sum_i \phi_i) \cdot \psi_j \cdot \omega = \sum_{j \in J} \int_M 1 \cdot \psi_j \cdot \omega = \overset{\psi}{\oint_M} \omega$.

Thus, $\overset{\phi}{\oint_M} \omega = \overset{\psi}{\oint_M} = \int_M \omega$.

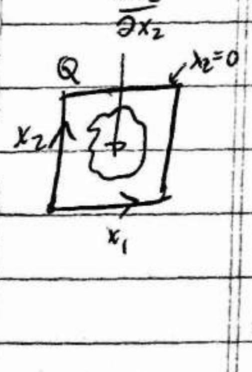
Properties: $a_i \in \mathbb{R}$

[1] $\int_M a_1 \omega_1 + a_2 \omega_2 = a_1 \int_M \omega_1 + a_2 \int_M \omega_2$

[2] $\int_{-M} \omega = -\int_M \omega = \int_M -\omega$.

<u>Stokes' Theorem</u> $\int_{\partial M} \omega = \int_M d\omega$. M^k compact & oriented, $\omega \in \Omega^{k-1}(M)$.

<u>CASE 1.</u> $\operatorname{supp} \omega \subseteq \operatorname{im} \alpha$, where $\alpha : Q \overset{+}{\to} M$ ($Q \subset \mathbb{R}^k$) s.t. $\alpha^* \omega \in \Omega^{k-1}(Q)$ and $\operatorname{supp} \alpha^* \omega \subseteq \operatorname{int} Q$. Let $\lambda = \alpha^* \omega$, since it doesn't depend on α.



Let $\lambda = \sum_{i=1}^k \lambda_i \, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_k$. Then: $\int_M d\omega = \int_Q \alpha^* d\omega = \int_Q d\lambda$

$\Longrightarrow \int_M d\omega = \sum_{i=1}^k (-1)^{i-1} \int_Q \frac{\partial \lambda_i}{\partial x_i} = 0 = \int_{\partial M} \omega$ since $\operatorname{supp} \omega \cap \partial M = \emptyset$.

Note that $d\lambda = \sum_{i=1}^k \frac{\partial \lambda_i}{\partial x_i} \cdot dx_i \wedge dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_k = \sum_i (-1)^{i-1} \frac{\partial \lambda_i}{\partial x_i} dx_K$

<u>CASE 2.</u> $\lambda \in \Omega^{k-1}(\underbrace{\mathbb{R}^k_{x_1 \geq 0}}_{= H_1^k})$ $\operatorname{supp} \lambda \subseteq \operatorname{int}_{H_1^k} Q$.

$Q = [0, b_1] \times \prod_{i=2}^k [a_i, b_i] = [0, b_1] \times Q'$.

$\longrightarrow$

$$\int_M d\omega = \int_Q d\lambda = \sum_{i=1}^{k} (-1)^{i-1} \int_Q \frac{\partial \lambda_i}{\partial x_i} = \int_Q \frac{\partial \lambda_1}{\partial x_1} = \int_{Q'} \int_0^{b_1} \frac{\partial \lambda_1}{\partial x_1} = \int_{Q'} \lambda_1 \Big|_0^{b_1}$$

$$= -\int_{x' \in Q'} \lambda_1(0,x') = -\int_{\{0\}\times Q'} \lambda_1\, dx_2 \wedge \cdots \wedge dx_k = -\int_{\{0\}\times Q'} \lambda = \int_{\partial Q} \lambda = \int_{\partial M} \omega.$$

<u>CASE 3.</u> $\omega = \sum \phi_i \omega$

$$\int_{\partial M} \omega = \int_{\partial M} \sum \phi_i \omega = \sum_i \int_{\partial M} \phi_i \omega = \sum_i \int_M d(\phi_i \cdot \omega) = \sum_i \int_M d\phi_i \wedge \omega + \phi_i \wedge d\omega$$

$$= \sum_i \int_M d\phi_i \wedge \omega + \sum_i \int_M \phi_i \wedge d\omega$$

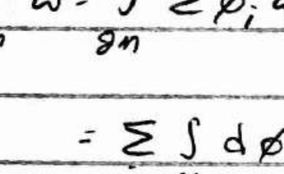
$$= \int_M \sum_i d\phi_i \wedge \omega + \int_M \Big(\sum_i \phi_i\Big) d\omega = \int_M d\Big(\sum \phi_i\Big) \wedge \omega + \int_M d\omega$$

$$= \int_M d(1) \wedge \omega + \int_M d\omega = 0 + \int_M d\omega = \int_M d\omega.$$

<u>Examples.</u> $\boxed{0}$ $M = [0,1]_x$ $\qquad$ $\boxed{1}$ $\omega \to f : [0,1] \to \mathbb{R}$

$\partial M = (+,1) \cup (-,0)$ $\qquad$ $\int_{[0,1]} f' = f(1) - f(0)$ $\qquad$ <u>FTC</u>

$\boxed{2}$



$\subseteq \mathbb{R}^2$ $\quad$ $\gamma : [0,1]_t \to \mathbb{R}^2$ s.t. $\gamma(0) = \gamma(1)$.

$\omega = P\,dx + Q\,dy \qquad P, Q : \mathbb{R}^2 \to \mathbb{R}$

$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$ $\qquad$ $d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_{\partial D = \mathrm{im}\gamma} P\,dx + Q\,dy = \int_{[0,1]} \gamma^*(P\,dx + Q\,dy)$$

$$= \int_{[0,1]} (P \circ \gamma) \cdot \dot{\gamma}_1\, dt + (Q \circ \gamma) \cdot \dot{\gamma}_2\, dt$$ Green's THM.

$$= \int_{[0,1]} (P \circ \gamma) \cdot \dot{\gamma}_1 + (Q \circ \gamma) \cdot \dot{\gamma}_2 = \int_{[0,1]} P \cdot \dot{\gamma}_1 + Q \cdot \dot{\gamma}_2 \to$$ line integral

$F = \begin{pmatrix} Q \\ -P \end{pmatrix}$ $\quad$ $F_1 = Q$, $F_2 = -P$ $\quad$ Two functions defines a vector field.

$$\int_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) = \int_D \operatorname{div} F \Rightarrow \int_{\partial D} (-F_2)\cdot\dot\gamma_1 + F_1\cdot\dot\gamma_2 = \int_{\partial D} \begin{pmatrix}F_1\\F_2\end{pmatrix}\cdot\begin{pmatrix}\dot\gamma_2\\-\dot\gamma_1\end{pmatrix}$$

$$= \int_{\partial D} \begin{pmatrix}F_1\\F_2\end{pmatrix}\cdot \underbrace{\begin{pmatrix}0&1\\-1&0\end{pmatrix}}_{90^\circ\text{ rot.}}\underbrace{\begin{pmatrix}\dot\gamma_1\\\dot\gamma_2\end{pmatrix}}_{\text{tangent to }\gamma} = \int_{\partial D} F\cdot\vec n\cdot\|\dot\gamma\| = \int_{[0,1]} F\cdot\vec n\cdot dV_{\partial D}.$$

Divergence Theorem. $\int_D \operatorname{div} F = \int_{[0,1]} F\cdot\vec n\cdot dV_{\partial D}$.

$\int$ fluid created at any point = total outflow.

Example. $\omega = \frac{1}{2}(x\,dy - y\,dx)$ $\quad d\omega = dx\wedge dy$.

$$\int_D d\omega = \int_D 1 = \text{Area}(D) \overset{\text{Stokes'}}{\underset{\text{THM}}{=}} \int_{\partial D}\omega.$$

area of domain by looking at bdry.

deRham complex	$\Omega^0(\mathbb{R}^3) \xrightarrow{d}$	$\Omega^1(\mathbb{R}^3) \xrightarrow{d}$	$\Omega^2(\mathbb{R}^3) \xrightarrow{d}$	$\Omega^3(\mathbb{R}^3)$
	$\{f\}$	$\{\sum a_i\,dx_i\}$	$\{b_1\,dx_{23} + \text{C.P.}\}$	$\{c\,dx_{\underline{3}}\}$
	functions	vector fields	vector fields	functions
vector calculus	$\{f\} \xrightarrow{\text{grad}}$	$\left\{\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\right\} \xrightarrow{\text{curl}}$	$\left\{\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\right\} \xrightarrow{\text{div}}$	$\{c\}$
manifolds	points with signs $U(s_i,p_i)$	curves $\Gamma = im(\gamma:[0,1]\to\mathbb{R}^3)$	surfaces $S = im(\sigma: D^2\to\mathbb{R}^3)$	solids/domains $D\subseteq\mathbb{R}^3$
integration.	$\int f = \sum s_i f(p_i)$ $\xrightarrow{\boxed{1}}$	$\int_\Gamma a\cdot\dot\gamma = \int_{\partial\Gamma} a\cdot\vec T\,dV$ $\xrightarrow{\boxed{2}}$	$\int_S b\cdot\dot\sigma = \int_{\partial S} b\cdot\vec n\,dV$ $\xrightarrow{\boxed{3}}$	$\int_D c = \int_{\partial D} c\cdot\vec n\,dV$.

Green's THM $\quad \overset{k=1}{\boxed{1}}$ curve Γ, function f

$$\int_\Gamma (\text{grad} f)\cdot\vec T\,dV = f(\gamma(1)) - f(\gamma(0))$$

integral of local climb rate $\quad$ elev. gain along a hike.

Stokes' THM $\quad \overset{k=2}{\boxed{2}}$ surface S, vector field a

$$\int_S (\text{curl } a)\cdot\vec n\,dV = \int_{\partial S} a\cdot\vec T\,dV$$

integral of local spinning in place of S $\quad$ circulation of a along ∂S.

Gauss' THM $\quad \overset{k=3}{\boxed{3}}$ domain D, vector field b

$$\int_D (\text{div } b)\,dV = \int_{\partial D} b\cdot\vec n\,dV$$

integral of local creation/annihilation of water flow $\quad$ total creation/annihilation of water flow along ∂D.

$\Omega^0(\mathbb{R}^3)$ $\quad \int_{p_i} f = \sum s_i f(p_i)$

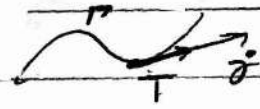
$\Omega^1(\mathbb{R}^3)$ $\quad a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \rightarrow w = \sum a_i \, dx_i \qquad \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$

unit tangent to T

$\dot\gamma = \|\dot\gamma\| \vec{T}$

$$\int_\Gamma w = \int_{[0,1]} \gamma^* w = \int_{[0,1]} \sum a_i dx_i = \int_{[0,1]} \sum a_i \dot\gamma_i \, dt$$

$$= \int_{[0,1]} a \bullet \dot\gamma_i = \int_{[0,1]} a \cdot \vec{T} \cdot \|\dot\gamma\| = \int a \cdot \vec{T} dV$$



$\Omega^2(\mathbb{R}^3)$ $\quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \rightsquigarrow w = b_1 \, dx_{23} + c.p. \qquad \sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$

$$\int_S w = \int_{D^2_{xy}} \sigma^* w = \int_{D^2} b \bullet \vec{n} \cdot V\left(\frac{\partial\sigma}{\partial x}, \frac{\partial\sigma}{\partial y}\right) = \int_S b \cdot \vec{n} \, dV.$$

$$\sigma^* w = \sigma^*(b_1 dx_2 \wedge dx_3 + c.p.) = b_1 d\sigma_2 \wedge d\sigma_3 + c.p.$$

$$= b_1 \left(\frac{\partial\sigma_2}{\partial x} dx + \frac{\partial\sigma_2}{\partial y} dy\right) \wedge \left(\frac{\partial\sigma_3}{\partial x} dx + \frac{\partial\sigma_3}{\partial y} dy\right) + c.p.$$

$$= b_1 \left(\frac{\partial\sigma_2}{\partial x} \cdot \frac{\partial\sigma_3}{\partial y} - \frac{\partial\sigma_3}{\partial x} \cdot \frac{\partial\sigma_2}{\partial y}\right) dx \wedge dy + c.p.$$

$$= \left(b_1 \left(\frac{\partial\sigma}{\partial x} \times \frac{\partial\sigma}{\partial y}\right)_1 + c.p.\right) dx \wedge dy = \sum b_i \left(\frac{\partial\sigma}{\partial x} \times \frac{\partial\sigma}{\partial y}\right)_i dx \wedge dy$$

$$= b \bullet \left(\frac{\partial\sigma}{\partial x} \times \frac{\partial\sigma}{\partial y}\right) dx \wedge dy = b \cdot \vec{n} \cdot V\left(\frac{\partial\sigma}{\partial x}, \frac{\partial\sigma}{\partial y}\right) dx \wedge dy.$$

$\Omega^3(\mathbb{R}^3) + \boxed{3}$ $\quad w_2(b) = b_1 dx_{23} + c.p.$, $\quad w_3(c) = c \, dx_{123}$ $\qquad \int_D \operatorname{div} b = \int_{\partial D} b \cdot \vec{n} \, dV$ $\Uparrow$

$$\int_D \operatorname{div} b = \int_D w_3(\operatorname{div} b) = \int_D dw_2(b) = \int_D dw = \int_{\partial D} w = \int_{\partial D} w_2(b) = \int_{\partial D} b \cdot \vec{n} \, dV$$

Let $w \in \Omega^k(M)$. $\boxed{1}$ w closed if $dw = 0$ $\rightarrow N^{k+1} \subset M \Rightarrow \int_{\partial N} w = \int_N dw = 0$

$\boxed{2}$ w exact if $\exists \lambda \in \Omega^{k-1}(M)$ such that $w = d\lambda$. $\rightarrow N^k \subset M$, $\partial N = \emptyset$ $\Rightarrow \int_N w = \int_N d\lambda = \int_{\partial N} \lambda = 0$.

$\Omega^{k-1} \xrightarrow{d} \Omega^k \xrightarrow{d} \Omega^{k+1}$ $\quad w$ closed $\Leftrightarrow w \in \ker d|_{\Omega^k}$

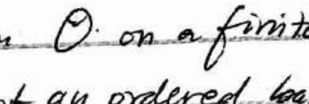
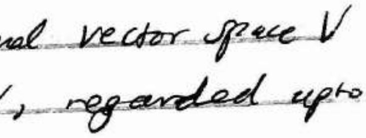
w exact $\Leftrightarrow w \in \operatorname{im} d|_{\Omega^{k-1}}$

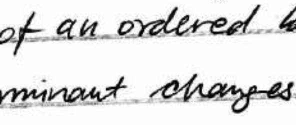
Every exact form is closed. $\quad w$ exact $\Rightarrow w = d\lambda \Rightarrow dw = 0$.

Definition. An orientation $\mathcal{O}$ on a finite dimensional vector space V is a choice of an ordered basis for V, regarded upto positive determinant changes of basis.

$(v_1 \dots v_n) \sim (u_1 \dots u_n)$ if $\det C_v^u > 0$ where $C_v^u = (c_{ij})$, $u_i = \sum c_{ij} v_j$.

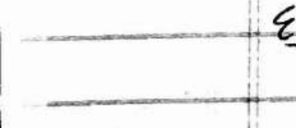
Examples. [3] $\mathbb{R}^3$, right-handed or left-handed.

[2] $\mathbb{R}^2$, ⓐ  [1] $\mathbb{R}$, ⓐ  ⓑ

ⓑ 

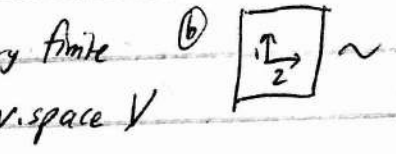
Claim. Every finite dimensional v.space V has 2 distinct orientations.

Definition. An orientation $\mathcal{O}$ on a k-dim manifold M is a continuous choice of an orientation $\mathcal{O}_x$ on T_xM $\forall x \in M$.



Every $p \in M$ has a neighborhood W with continuous vector fields $u_1 \dots u_k$ defined on W, such that $\forall x \in W: (u_1(x) \dots u_k(x)) \sim \mathcal{O}_x$

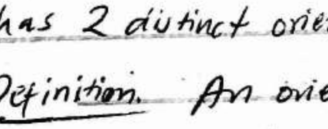
Examples. [1]



[2]

HAIRY BALL
Every V.F. on S^2 has at least one zero. "cowlick"

[3]



no way to find an orientation on a mobius band (get two orientations by going around).

If $M^k \subset \mathbb{R}^{k+1}$ and $\mathbb{R}^{k+1}$ is oriented, then there is a bijection:

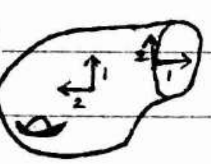
$$\left\{ \begin{matrix} \text{orientations} \\ \text{of } M \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{continuous varying choice} \\ \text{of unit normal vectors to} \\ M \text{ in } \mathbb{R}^{k+1} \end{matrix} \right\}$$

$$\Updownarrow$$

$\forall p \in M$ $n(p) \in T_p\mathbb{R}^{k+1}$ such that: ① $n(p) \perp T_pM$ ② $\|n(p)\| = 1$
③ $p \mapsto n(p)$ is continuous.

Claim. If M^k is orientable, then so is ∂M.

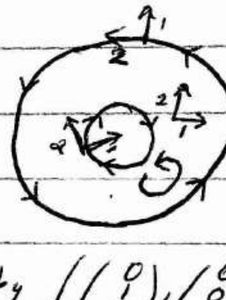
"∂M's orientation is such that prepending to it the outward normal of ∂M gives M's orientation."

Examples. [1] 

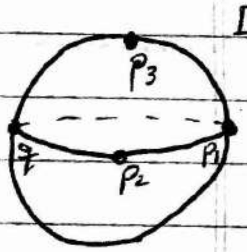
[2] $A = \{x \in \mathbb{R}^2 : 1 \le \|x\| \le 2\}$ std. $\mathcal{O}: (e_1, e_2)$.

Outer-circle $\to$ counter-clockwise

inner-circle $\to$ clockwise



[3] $D^3 = \{x \in \mathbb{R}^3 : \|x\| \le 1\} \subseteq \mathbb{R}^3$ std $\mathcal{O}: (e_1, e_2, e_3)$.



$\partial D^3 = S^2$ – How is S^2 oriented at p_i, i.e. how is $T_{p_i} S^2$ oriented?

$p_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, p_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, p_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$T_{p_1} S^2$ oriented by $\left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$

$T_{p_2} S^2$ oriented by $\left(\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right)$

$T_{p_3} S^2$ oriented by $\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$

$T_q S^2$ oriented by $\left(\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$.

$(-e_1, e_3, e_2) \sim (e_1, e_2, e_3)$.

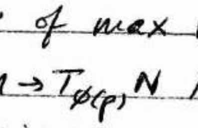
If $T: V \to W$ an isomorphism of an oriented vector space, then T is:

"orientation preserving" $\to$ pushes a basis agreeing w/ orientation of V to a basis agreeing w/ orientation of W.

"orientation reversing" $\to$ otherwise

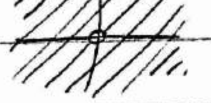
If $\phi: M^k \to N^k$ is a C^r map, whose differential is of max rank then ϕ is: "orientation preserving" $\to + \to D\phi_p : T_pM \to T_{\phi(p)}N$ is "orientation preserving"

"orientation reversing" $\to$ otherwise

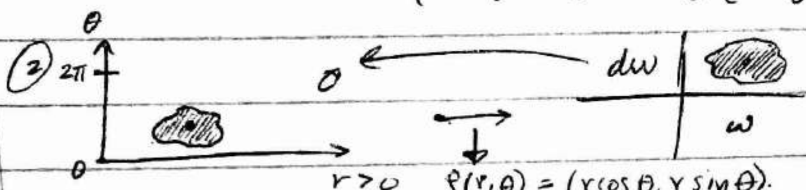
The composition of 2 positive maps is positive: 

If M is oriented, α & β are positive & $\operatorname{supp} \omega \subseteq \operatorname{im} \alpha, \operatorname{im} \beta$, then $\int_{\mathbb{R}^k} \alpha^* \omega = {}^{\alpha}\!\int_M \omega = {}^{\beta}\!\int_M \omega = \int_{\mathbb{R}^k} \beta^* \omega =: \int_M \omega$.

$\hookrightarrow$ Only makes sense when M is oriented.

Example. $\omega = \frac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$  punctured plane

ω closed: ① $d\omega = \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right) dx \wedge dy = 0$

② 

$r > 0$ $\quad \rho(r,\theta) = (r\cos\theta, r\sin\theta)$.

$\rho^*(d\omega) = d\rho^*\omega = d\left(\frac{r^2(\cos^2\theta + \sin^2\theta)}{r^2}\, d\theta\right) = d(d\theta) = 0.$

(pullback of ρ closed) $\Rightarrow d\omega = 0$ since nr. every pt $p \in \mathbb{R}^2_{r,\theta}$, ρ is invertible.

$\omega = d\left(\arctan \frac{y}{x}\right) \Rightarrow d\omega = d\left(d\left(\arctan\frac{y}{x}\right)\right) = 0$ except if $x = 0$.

$= d\left(\arctan\left(-\frac{x}{y}\right) \pm \frac{\pi}{2}\right)$ $\qquad$ not fully-defined $(x \neq 0)$

ω is not exact (but nearly so). $\therefore\ \omega = d\left(\arctan\frac{y}{x}\right)$ $(x \neq 0)$.

$\Downarrow$

$\int_{S^1} \omega = \int_{S^1} x\,dy - y\,dx = 2\pi \neq 0. \Rightarrow \omega$ is not exact.

Poincaré's Lemma. On $\mathbb{R}^n$, every closed form is exact.

de-Rham. If M is compact, closed $\xRightarrow{\text{nearly}}$ exact.