Feb. Wed 1st Wed Hour ~~20~~ 052

Read along: 29-31

K-form on M: $W: M \to \bigcup_{p \in M} A^k(T_pM)$,

Such that $W(p) \in A^k(T_pM)$ on $\mathbb{R}^n$

$$W = \sum_{I \in \binom{n}{k}} a_I(x)\psi_I = \sum_I a_I(x)\phi_{i_1} \wedge \phi_{i_2} \wedge \cdots \wedge \phi_{i_k}$$

W is C^r means $\forall I$, a_I is $C^r \Leftrightarrow \forall C^r$ v.f. $Y_1, \cdots, Y_k$ (+ on tangent vectors)

$W(Y_1, \cdots, Y_k)$ is C^r. Forms pull back!

$F: \mathbb{R}^n \to \mathbb{R}^m$. w on $\mathbb{R}^m$. $(F^* w)(\xi_1, \cdots, \xi_k) = W(F^*\xi_1, \cdots, F^*\xi_k)$

$$F^* w \longrightarrow w$$

Def: $\Omega^k(\mathbb{R}^n)$ ($\Omega^k(M)$): the set of all C^∞ k-forms on $\mathbb{R}^n/M$

$(W_1 + W_2)(\xi_1, \cdots, \xi_k) = W_1(\xi_1, \cdots, \xi_k) + W_2(\xi_1, \cdots, \xi_k)$

$\lambda(w)(\sim) = w(\lambda\xi_1, \cdots, \lambda\xi_k)$

$\wedge: A^k(V) \times A^l(V) \to A^{k+l}(V)$

Def: $w \in \Omega^k(M)$, $\eta \in \Omega^l(M)$

$\Omega^{k+l}(M)$ $\quad \xi_i = (x, v_i)$, $v_i \in T_xM$

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$(w \wedge \eta)(\xi_1, \cdots, \xi_{k+l}) = (W(x) \wedge \eta(x))(v_1, \cdots, v_{k+l})$

Properties:
1. Bilinear and associative
2. Super-Symmetric: $w \wedge \eta = (-1)^{k+l}\, \eta \wedge w$
3. $F^*(w \wedge \eta) = (F^* w) \wedge (F^* \eta)$

On

$$\mathbb{R}^3 \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3)$$

$\Omega^0(\mathbb{R}^3)$?

$$\left\{\begin{matrix} a_1(x)\phi_1 \\ +a_2(x)\phi_2 \\ +a_3(x)\phi_3 \end{matrix}\right\} \quad \left\{\begin{matrix} b_1(x)\phi_2\wedge\phi_3 \\ +b_2(x)\phi_3\wedge\phi_1 \\ +b_3(x)\phi_1\wedge\phi_2 \end{matrix}\right\} \quad \{c(x)\phi_1\wedge\phi_2\wedge\phi_3\}$$

$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Updownarrow$$

$$a(x) = \begin{pmatrix} a_1(x) \\ a_2(x) \\ a_3(x) \end{pmatrix} \qquad b(x) = \begin{pmatrix} b_1(x) \\ b_2(x) \\ b_3(x) \end{pmatrix} \qquad c(x)$$

$$\text{III} \qquad\qquad \text{III} \qquad\qquad \text{III}$$

Vector field $\qquad$ v.f. $\qquad$ functions

$\rightarrow$ 0-form on M: $W: M \to \bigcup_{p\in M} \underbrace{A^*(T_pM)}_{\mathbb{R}}$

$W: M \to \mathbb{R} \Leftrightarrow$ a function on M

W is a 0-form, and η is a k-form

$$\underbrace{(W\wedge\eta)}_{k\text{-form}}(\xi_1,\cdots,\xi_k) = W(x)\cdot\eta(x)(v_1,\cdots,v_k)$$

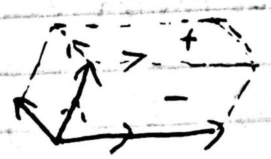
So $\Omega^0(\mathbb{R}^3) = \{F(x)\} \leftrightarrow$ functions

$d: \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$, "right def^{n}" (but too hard):

Suppose $W \in \Omega^k(\mathbb{R}^n)$, $(dW)(\xi_1,\cdots,\xi_{k+1}) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{k+1}} W(\partial p(\varepsilon\xi_1,\cdots,\varepsilon\xi_{k+1}))$

$\uparrow$ big sum with signs of evaluating of w on k vectors.

Ex: $k=2$



$\to P(\xi_1,\cdots,\xi_3)$
$2(k+1)$ faces

∂p: Union of $2(k+1)$ faces. $\quad \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$

following def: Let $F \in \Omega^0(\mathbb{R}^n)$, ($F$ is a function)

Ex:

What is $(dF)(\xi) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(F(x+\varepsilon v) - F(x) \right) = D_\xi F$

$\uparrow$

$\Omega^1(\mathbb{R}^n)$, $\xi = (x, v)$

$\varepsilon\xi$ $+$

$-$ x $\quad x+v \; (x+\varepsilon v)$