MAT257 Midterm 2

$f:\mathbb{R}^n\to\mathbb{R}$ bounded, $Q=\prod_{j=1}^{n}[a_j,b_j]$, Partition of Q : $\underline{P}=(P_1,\dots,P_n)$ where $P_j=(a_j=t_{j_0}<t_{j_1}<\dots<t_{j_{k_j}}=b_j)$ partition of $[a_j,b_j]$.

Rectangle $R\in\underline{P} \Leftrightarrow R=\prod_{j=1}^{n}[c_j,d_j]$ such that $\forall j.\,[c_j,d_j]\in P_j$.

$V(R):=\prod_{j=1}^{n}(d_j-c_j)$, is the volume of Rectangle R.

$$m_R(f)=\inf_{x\in R} f(x) \qquad L(f,\underline{P})=\sum_{R\in\underline{P}} m_R(f)\cdot V(R) \qquad \underline{\int_Q} f=\sup_{\underline{P}} L(f,\underline{P})$$
$$\leq \qquad\qquad \leq$$
$$M_R(f)=\sup_{x\in R} f(x) \qquad U(f,\underline{P})=\sum_{R\in\underline{P}} M_R(f)\cdot V(R) \qquad \overline{\int_Q} f=\inf_{\underline{P}} U(f,\underline{P}).$$

f is integrable on $Q \Leftrightarrow \underline{\int_Q} f=\overline{\int_Q} f=:\int_Q f$

$P'=(a=t_0'<t_1'<\dots<t_{k'}'=b)$ is a refinement of $P=(a=t_0<t_1<\dots<t_k=b)$ if: $\forall j.\ t_j\in P'$ ($\forall j\ \exists j'.\ t_j=t_{j'}'$). $\underline{P}'=(P_1',P_2',\dots,P_n')$ refines $\underline{P}=(P_1,P_2,\dots,P_n)$ if $\forall j.\ P_j'$ is a refinement of P_j.

Lemma. If $\underline{P}'$ refines $\underline{P}$, then $L(f,\underline{P}')\geq L(f,\underline{P})$ and $U(f,\underline{P}')\leq U(f,\underline{P})$.

Lemma. For any $\underline{P}$ and $\underline{P}'$, $L(f,\underline{P})\leq U(f,\underline{P}')$.

THM. f is integrable IFF $\forall\epsilon>0.\ \exists\underline{P}$ s.t. $U(f,\underline{P})-L(f,\underline{P})<\epsilon$.

THM. Every continuous function on Q is integrable.

f is continuous on Q: $\forall x\in Q.\ \forall\epsilon>0.\ \exists\delta>0.\ \forall y\in Q.\ |x-y|<\delta\Rightarrow|f(x)-f(y)|<\epsilon$.

f is uniformly continuous on Q: $\forall\epsilon>0.\ \exists\delta>0.\ \forall x\in Q.\ \forall y\in Q.\ |x-y|<\delta\Rightarrow|f(x)-f(y)|<\epsilon$.

THM. Every continuous function on a compact set is uniformly continuous.
THM. Every uniformly continuous function is integrable.
↳ THM. Every continuous function on Q is integrable.

Lemma. If $\{U_\alpha\}$ is an open cover of a compact space (X,d), then there exists $\delta>0$ (called the Lebesgue number of the cover $\{U_\alpha\}$) such that any ball $B = U(x,\delta)$ is contained in at least one of the sets U_α.

THM. A bounded function $f: Q \to \mathbb{R}$ ($Q \subset \mathbb{R}^n$) is integrable IFF the set of discontinuities of f is of measure 0.

Let $f: X \to Y$. The disco-set of f: $D = D(f) = \{x \in X : f \text{ is not continuous at } x\}$.

A set $A \subset \mathbb{R}^n$ is of measure 0 if for every $\epsilon>0$, there exists a countable collection R_i of rectangles in $\mathbb{R}^n$, i.e. $\exists$ rectangles $\to \{R_i\}_{i=1}^{\infty}$ such that:
① $A \subset \bigcup_{i=1}^{\infty} R_i$ ② $\sum_{i=1}^{\infty} v(R_i) < \epsilon$.

THM. If A is meas-0, then $B \subset A$ is meas-0.
THM. A countable unions of meas-0 sets is meas-0.
THM. Defn. of meas-0 set A doesn't change when $A \subset \bigcup_{i=1}^{\infty} R_i \to A \subset \bigcup_{i=1}^{\infty} \operatorname{int} R_i$

$f: Q \to \mathbb{R}$ integrable. ① f almost always 0 ($\{x \in Q : f(x) \neq 0\}$ is meas-0) $\Rightarrow \int_Q f = 0$.
② $f \geq 0$ and $\{x \in Q : f(x) > 0\}$ is not meas-0 $\Rightarrow \int_Q f > 0$.

If f is continuous on $[a,b]$ then: ① $F(x) = \int_a^x f$, then F' exists and $F' = f$.
① If g is such that $g' = f$, then $\int_a^b f = g(b) - g(a)$.
$(a = t_0 < \dots < t_k = b)$ $\sum f(t_i) \sim \sum_i g(t_i) - g(t_{i-1}) \Rightarrow g(b) - g(a)$, by telescopic summation.

THM. $A \subset \mathbb{R}^n$ a rectangle. $B \subset \mathbb{R}^m$ a rectangle. $Q = A \times B \subset \mathbb{R}^{n+m}$.
Suppose $f: Q \to \mathbb{R}$ is integrable. Define $l(x) = \underline{\int}_{\{x\}\times B} f$ and $u(x) = \overline{\int}_{\{x\}\times B} f$.
Then l and u both integrable on A and $\int_A l = \int_Q f = \int_A u$.

Corollary. If for every $x \in A$ $f(x,-)$ is integrable over B ($f(x,y)$ is integrable relative to $y \in B$), then $\int_{A\times B} f = \int_{x\in A} \int_{y \in B} f(x,y)$.

Corollary. $\int_{A\times B} f = \int_{x\in A}\int_{y\in B} f(x,y) \overset{\text{Fub.}}{=} \int_{y\in B}\int_{x\in A} f(x,y)$, given all integrals exist.

Corollary. If all integrals exist, $Q = \prod [a_i, b_i]$, then:
$$\int_Q f = \int_{x_1\in[a_1,b_1]} \int_{x_2\in[a_2,b_2]} \cdots \int_{x_n \in [a_n,b_n]} f(x_1,\dots,x_n).$$

If $S \subset \mathbb{R}^n$ is bounded and f is integrable on S (i.e. $\int_Q f 1_S$ exists) and $\int_S f := \int_Q f 1_S$, where $1_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$, and Q is some rectangle containing S. (Doesn't matter on choice of Q).

THM. [1] f, g integrable over S, $\int_S af+bg = a\int_S f + b\int_S g$, $a,b \in \mathbb{R}$ arbitrary.
[2] f,g integrable over S, $f \le g \Rightarrow \int_S f \le \int_S g$.
[3] f integrable over $S \Rightarrow |\int_S f| \le \int_S |f|$.
[4] $f \ge 0$ integrable, $T \subseteq S \Rightarrow \int_T f \le \int_S f$.
[5] $\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f - \int_{S_1 \cap S_2} f$, given f integrable over S_1, S_2.
If $S_1 \cap S_2$ meas-0, then $\int_{S_1\cup S_2} f = \int_{S_1} f + \int_{S_2} f$.

$S \subset \mathbb{R}^n$ is rectifiable if 1_S is integrable and $V(S) := \int_S 1 = \int_Q 1_S$.

THM. A set S is rectifiable IFF S is bounded and $Bd(S)$ is meas-0.

THM. [1] $V(S) \geq 0$ Pf. $1_S \geq 0$

[2] $S_1 \subset S_2 \Rightarrow V(S_1) \leq V(S_2)$.   Pf. $1_{S_1} \leq 1_{S_2}$

[3] If S_1, S_2 rectifiable, then so are $S_1 \cup S_2$ and $S_1 \cap S_2$ and

$$V(S_1 \cup S_2) = V(S_1) + V(S_2) - V(S_1 \cap S_2).$$

[4] If S is rectifiable, $V(S) = 0 \Leftrightarrow S$ is measure-0.

[5] If S is rectifiable and $f: S \to \mathbb{R}$ is bounded and continuous, then $\int_S f$ makes sense.

THM. If C is a compact rectifiable set in $\mathbb{R}^n$ and $f, g: C \to \mathbb{R}$ are continuous and $f \leq g$ on C, then $D = \{(x,t) : x \in C, f(x) \leq t \leq g(x)\} \subset \mathbb{R}^{n+1}$ is rectifiable, $V(D) = \int g \circ f$, if $h: D \to \mathbb{R}$ is continuous, then $\int_D h = \int_{x \in C} \int_{f(t)}^{g(t)} h(x,t)$.

Note: D is rectifiable.

Claim. $Bd(D) = \{(x, f(x)) : x \in C\} \cup \{(x, g(x)) : x \in C\} \cup \{(x,t) : x \in Bd\, C.\ f(x) \leq t \leq g(x)\}$.

Assume $A \subseteq \mathbb{R}^k$, $f: A \to \mathbb{R}$ continuous. $f_+ = \max(f, 0)$ $f = f_+ - f_-$
$f_- = \max(-f, 0)$ $|f| = f_+ + f_-$

Note that f_+, f_- both continuous since max, composition continuous.

$f \geq 0 \Rightarrow \fint_A f := \sup \left\{ \fint_D f : \substack{D \subset A \\ \text{compact, rectifiable}} \right\}$, when sensible.

$f \leq 0 \Rightarrow \fint_A f := \fint_A f_+ - \fint_A f_-$, when sensible.

$C_n \nearrow A$ (C_n rises to A) if: [1] C_n compact, rectifiable set
[2] $C_n \subset \operatorname{int} C_{n+1}$ [3] $\bigcup_{n=1}^{\infty} C_n = A$.

THM. Given $C_n \nearrow A$, $\fint_A f$ exists IFF $\fint_{C_n} |f|$ is bounded and then, $\fint_A f = \lim_{n \to \infty} \fint_{C_n} f$.

Claim. For any A, one can find C_n such that $C_n \nearrow A \subset \mathbb{R}^k$.

$\underline{\text{THM}}$. 1 ${}^{E}\!\!\int_A af+bg = a\,{}^{E}\!\!\int_A f + b\,{}^{E}\!\!\int_A g$

2 $f \le g \Rightarrow {}^{E}\!\!\int_A f \le {}^{E}\!\!\int_A g$ & ${}^{E}\!\!\int_A f \le {}^{E}\!\!\int_A |f|$ & ${}^{E}\!\!\int_A -f \le {}^{E}\!\!\int_A |f| \Rightarrow \left|{}^{E}\!\!\int_A f\right| \le {}^{E}\!\!\int_A |f|$.

3 If $B \subset A$, ${}^{E}\!\!\int_B f \le {}^{E}\!\!\int_A f$, given that $f \geq 0$.

4 If A, B both open & f integrable on both $\Rightarrow {}^{E}\!\!\int_{A\cup B} f = {}^{E}\!\!\int_A f + {}^{E}\!\!\int_B f - {}^{E}\!\!\int_{A\cap B} f$.

$\underline{\text{THM}}$. If $A \subset \mathbb{R}^n$ is bounded and open and $f: A \to \mathbb{R}$ is bounded and continuous, then ${}^{E}\!\!\int_A f$ exists and if ${}^{R}\!\!\int_A f$ also exists, then ${}^{E}\!\!\int_A f = {}^{R}\!\!\int_A f$.

$\underline{\text{THM}}$. If $A \subset \mathbb{R}^k$ is open, $f: A \to \mathbb{R}$ continuous and $U_1 \subset U_2 \subset U_3 \subset \dots$ are open and $\bigcup_{i=1}^{\infty} U_n = A$, then ${}^{E}\!\!\int_A f$ exists IFF ${}^{E}\!\!\int_{U_n} |f|$ is bounded and then ${}^{E}\!\!\int_A f = \lim_{n\to\infty} {}^{E}\!\!\int_{U_n} f$.

Let $g: A \to B$ be a diffeomorphism $\left(\begin{smallmatrix}\text{1-1, onto} \\ C^1, C^1 \text{ inverse}\end{smallmatrix}\right)$ of open sets in $\mathbb{R}^n$. Let $f: B \to \mathbb{R}$. Then f is integrable on B IFF $(f \circ g)\cdot J_g$ is integrable on A, $J_g(x) := |\det Dg(x)|$, the Jacobian of g at x, and

$$\int_B f = \int_A (f\circ g)\,|\det Dg| = \int_A (f\circ g)\cdot J_g .$$

$$\begin{tikzcd} \underset{\subset \mathbb{R}^n}{\overset{\cdot x}{A \text{ open}}} \arrow[r, "g", "{\substack{\text{1-1,} \\ \text{onto,} \\ C^1, C^1 \text{inverse}}}"'] \arrow[rr, bend left, "(f\circ g)"] & \underset{\subset \mathbb{R}^n}{B \text{ open}} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

Key Lemma. If $L: \mathbb{R}^n \to \mathbb{R}^n$ is linear and P' is a parallelogram in $\mathbb{R}^n$, then:

$$V(L(P')) = |\det(L)\cdot V(P')|.$$

$\underline{\text{THM}}$. Given $A \xrightarrow{g} B \xrightarrow{f} \mathbb{R}$, $\int_B f = \int_A (f\circ g)\cdot|\det Dg|$.

Lemma. THM true for $g_1, g_2 \Rightarrow$ true for $g_1 \circ g_2$.

Lemma. THM true for affine-linear maps: $g(x) = Ax+b$, $A \in M_{n\times n}$ invertible, $b \in \mathbb{R}^n$.

① $g(x) = x+b$ ② Coord swaps ③ coord rescales ④ coord add multiple.

$|J| = 1$ $\left(\begin{smallmatrix}1&&\\&0\,1&\\&1\,0&\\&&1\end{smallmatrix}\right)$ $|J| = 1$ $\left(\begin{smallmatrix}1&&\\&c&\\&&1\end{smallmatrix}\right)$ $|J| = |c|$ $\left(\begin{smallmatrix}1&c\\0&1\end{smallmatrix}\right)$ $|J| = 1$

Suppose $v_1, \dots, v_n \in \mathbb{R}^n$. $P(v_1, \dots v_n)$ = "parallelpiped spanned by $v_1, \dots, v_n$," i.e. $P(v_1, \dots, v_n) = \{ \sum a_i v_i : 0 \leq a_i \leq 1, \forall i \}$ and $V(P(v_1, \dots v_n)) = |\det(v_1 | \dots | v_n)|$.

$h: \mathbb{R}^n \to \mathbb{R}^n$ is called an isometry if $\forall x, y$ $d(h(x), h(y)) = d(x, y)$.

THM. $h: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry IFF it can be written in the form $h(x) = p + Ax$ where $p \in \mathbb{R}^n$, $A \in M_{n \times n}$ such that $\underbrace{A^T \cdot A = I}_{A \text{ is orthogonal.}}$. Such an h is volume-preserving.

$A = \{v_i\}$ forms an orthonormal basis.

Rotation matrices / orthogonal matrices $O(n) = \{ A \in M_{n \times n}(\mathbb{R}) : A^T A = I \}$ form a group:

1. $A, B \in O(n) \Rightarrow AB \in O(n)$, $A(BC) = (AB)C$.
2. $\exists I \in O(n)$ such that $AI = IA = A$ $\forall A \in O(n)$.
3. $\forall A \in O(n)$ $\exists B \in O(n)$ such that $AB = BA = I$.

($\Leftarrow$) Show $d(h(x), h(y)) = d(x, y)$.

($\Rightarrow$)
1. WLOG, $h(0) = 0$
2. h preserves norms
3. h preserves inner products.
4. $A = (h(e_1) | h(e_2) | \dots | h(e_n)) \in O(n)$
5. h is linear

Gram-Schmidt.

$v_1' = u_1$ $\quad v_1 = \pm v_1' / \|v_1'\|$

$v_2' = u_1 - \langle u_1, v_1 \rangle v_1$ $\quad v_2 = \pm v_2' / \|v_2'\|$

$\vdots$

$v_k' = u_k - \sum_{j=1}^{k-1} \langle u_k, v_j \rangle v_j$ $\quad v_k = \pm v_k' / \|v_k'\|$.

THM. If $A^T A = I$, $A \in M_{n \times n}(\mathbb{R})$, $V(v_1 \cdots v_k) = V(Av_1, \dots Av_k)$.

THM. There's a unique $V: (\mathbb{R}^n)^k \to \mathbb{R}_{\geq 0}$ such that:

① If $h: \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation & $x_i \in \mathbb{R}$, $V(h(x_1) \dots h(x_k)) = V(x_1, \dots x_k)$.

② If $x_i \in \mathbb{R}^k \times \{0\}$, so $x_i = \begin{pmatrix} y_i \\ 0 \end{pmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix}$ w/ $y_i \in \mathbb{R}^k$, then $V(x_1, \dots x_k) = |\det(y_1 | \dots | y_k)|$.

③ $V(x_1, \dots x_k) = 0 \iff \{x_i\}$ is dependent. $\hookrightarrow$ V exists and is unique.

④ $X = (x_1 | \dots | x_k) \in M_{n \times k} \Rightarrow V(x_1, \dots x_k) = |\det(X^T \cdot X)|^{1/2}$.