MAT240 - Tutorial Sept. 28, 2006

<u>Claim:</u> In $\mathbb{R}^{\cancel{2}3} = \{(a,b) \mid a,b \in \mathbb{R}\}$ (intuitively the plain)

$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ (adding vectors)

$r(a,b) = (ra, rb)$ (dilation or contraction)

<u>The vector spaces contained in $\mathbb{R}^2$ are:</u> $\Rightarrow$ (subspaces in $\mathbb{R}$)

$\{0\}$, $\mathbb{R}^{\cancel{2}3}$, lines passing through origin $(0,0)$, planes passing through origin.

S is a subspace,

(1) $0 \in S$

(2) $v + w \in S$ if $v, w \in S$

(3) $rv \in S$ if $r \in \mathbb{R}$, $s \in S$

<u>Claim:</u> A line, L, passing through origin is a subspace.

Pick $v \in L$, $v \neq (0,0)$

$L = \{\lambda v \mid \lambda \in \mathbb{R}\}$

(1) $\lambda = 0 \Rightarrow 0 \in L$.

(2) $\lambda_1 v + \lambda_2 v = (\lambda_1 + \lambda_2) v \in L$.

(3) easy

<u>Exercise:</u> planes passing through origin are subspaces.

$\boxed{A2}$ * for T/F Q's: have to prove smthg is true, or give counter ex it's false.

1 (a) T

b) T

18. $V = \{(a_1, a_2) \mid a_i \in \mathbb{R}\}$

$$(a_1, a_2) = (a_1 + 2b_1, a_2 + 3b_2)$$
$$c(a_1, a_2) = (ca_1, ca_2).$$

↪ not comm. in addition.

$$(b_1, b_2) + (a_1, a_2) = (b_1 + 2a_1, b_2 + 3a_2)$$
$$1 \qquad 0$$

Give counter example to show it's false

$$(1,0) + (0,0) = (1,0)$$
$$(0,0) + (1,0) = (2,0).$$

19. $V = \{(a_1, a_2) \mid a_i \in \mathbb{R}\}.$

$$c(a_1, a_2) = \begin{cases} (ca_1, \frac{a_2}{c}) & c \neq 0 \\ (0,0) & c = 0. \end{cases}$$

(VS8) $(a+b)x = ax + bx$

$$(1+1)(a_1, a_2) = (a_1, a_2) + (a_1, a_2)$$
$$2(a_1, a_2) \qquad (2a_1, 2a_2)$$
$$\left(2a_1, \frac{a_2}{2}\right) \quad \underline{eg}\ a_2 = 1.$$

They are different ∴ counterex

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8. (a) $\{(a_1, a_2, a_3) \mid a_1 = 3a_2,\ a_3 = -a_2\}$. $\Rightarrow$ Subspace of $\mathbb{R}^3$?

$= \{(3a_2, a_2, a_2) \mid a_2 \in \mathbb{R}\}$.

$= \{a_2(3,1,1) \mid a_2 \in \mathbb{R}\}$. $\Rightarrow$ it's a line $\therefore$ subspace.

But in assignment, have to go thru all 3 axioms.

(b) $\{(a_1, a_2, a_3) \mid a_1 = a_3 + 2\}$.

$(0,0,0)$ is not in here $\therefore$ it's not subspace.

* keep proofs as concise as possible.

(c) $\{(a_1, a_2, a_3) \mid 2a_1 - 7a_2 + a_3 = 0\}$.

$= \{(a_1, a_2, -2a_1 + 7a_2)\}$

$= \{a_1(1,0,-2) + a_2(0,1,7)\}$ $\Rightarrow$ it's a plane $\therefore$ subspace.

(d) same as c)

(e) same as b)

(f) $\{(a_1, a_2, a_3) \mid 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$.

$0 \in W_4$. ✓

violates addition:

$(0, \sqrt{2}, 1) + (0, -\sqrt{2}, 1)$

$= (0, 0, 2) \notin W_4$.

~~$2(0, \sqrt{2}, 1)$~~

~~$= (0, 2\sqrt{2}, 2)$~~

~~$5 \cdot 0^2 - 3(2\sqrt{3})^2 + 6(2)^2$~~

~~$=$~~

9. $W_1 = \{a_1 = 3a_2, a_3 = -a_2\}$. $W_3 = \{2a_1 - 7a_2 + a_3 = 0\}$.

$W_1 \cap W_3 = \{(a_1, a_2 a_3) | = a_1 = 3a_2, a_3 = -a_2, 2a_1 - 7a_2 + a_3 = 0\}$

$= \{(3a_2, a_2, -a_2) | 2(3a_2) - 7a_2 - a_2 = 0\} \in$

$= \{(3a_2, a_2, -a_2) | -3a_2 = 0\}$

$= \{(0,0,0)\}$.

11. $W = \{f(x) \in P(F) | f(x) = 0 \text{ or } f(x) \text{ deg } n\}$, $n \geq 1$.

Case (1) $n = 1$ yes

Case (2) $n > 1$ no.

$\cancel{a_n x^n + a_{n-1} x^{n}}$

$x^n + x^{n-1} + \dots + a_0$

$+ -x^n + x^{n-1} + \dots + a_0$

$0 + 2x^{n-1} + \dots + a_0$

eg $n = 3$ $x^3 + x^2 (-x^3 + x^2) = 2x^2$

not polynomial of degree n anymore, so doesn't satisfy rule.

$V = \{f(x) \in P(F) | \deg f \leq n\}$ is a subspace

(1) $0 \in V$

(2) $\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i$ $a_n, b_n \neq 0$

$= \sum_{i=0}^{n} (a_i x^i + b_i x^i)$

$= \sum_{i=0}^{n} (a_i + b_i) x^i \in V$

(3) $r\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} (r a_i) x^i \in V$. by distributive property

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19. $W_1 \cup W_2$ subspace $\iff W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Pf. $W_1 \subseteq W_2 \Rightarrow W_1 \cup W_2 = W_2$ $\therefore W_1 \cup W_2$ is subspace

($\Leftarrow$)

$W_2 \subseteq W_1 \Rightarrow W_1 \cup W_2 = W_1$ $\therefore W_1 \cup W_2$ is subspace

($\Rightarrow$) If $u, v \in W_1 \cup W_2$ ($\because W_1 \cup W_2$ is a subspace)

$\Rightarrow u+v \in W_1$ or $u+v \in W_2$.

If $u+v \in W_1$

$\vdots$

Using contradiction

Suppose $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$

$\Rightarrow \exists\, u \neq 0 \in W_2 \setminus W_1$ and $\exists\, u \neq 0 \in W_1 \setminus W_2$

$\Rightarrow u, v \in W_1 \cup W_2$

$\Rightarrow u+v \in W_1 \cup W_2$ ($\because W_1 \cup W_2$ subspace)

~~But $u \notin W_1 \cup W_2$ $\Rightarrow -u \in W_1 \cup W_2$ ($W_1 \cup W_2$ is a subspace)~~

$\Rightarrow u+v \in W_1$ or W_2

Suppose $u+v \in W_1$ but $v \in W_1 \Rightarrow -v \in W_1$ (W_1 is subspace)

$\underbrace{(u+v)+(-v)}_{= u} \in W_1$ (W_1 subsp)

Contradiction $\because u \notin W_1$

So $u+v \in W_2$, similarly show $v \in W_2$ and get contradiction question on sheet:

$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, p prime

(1) $b := 1 \cdot 2 \cdots (p-1) \neq 0$ b/c $1 \cdot 2 \cdots (p-1) = 0 \pmod p$
$\cdot$ $1 \cdot 2 \cdots (p-1)$ is dividible by p.
Contradiction $\because$ p is prime.

(2) $\{1, 2, \dots, p-1\} = \{a, 2a, \dots, (p-1)a\}$, $a \not\equiv 0 \pmod p$

Suppose not, $ma = na$ for some $0 < m, n < p$.
~~$m\,a\,a^{-1} \pmod p$~~
Let b be an integer s.t. $ab = 1 \pmod p$

$\Rightarrow m \cdot a \cdot b = n \cdot a \cdot b$
$\Rightarrow m = n \pmod p$
$\Rightarrow m = n$

(3) $b a^{p-1} \overset{?}{=} b$ $b := 1 \cdots (p-1)$

~~$1 \cdots (p-1)$~~ $\pmod p$
$\cdot 1 \cdots (p-1) a^{p-1} = 1 \cdots (p-1)$
$(1 \cdot a)(2 \cdot a) \cdots ((p-1) \cdot a) = \text{LHS}$

$b \neq 0 \pmod p$
$\Rightarrow a^{p-1} = 1 \pmod p$ $(*)$

(4) $4^6 - 1$ divisible by 7.
$4^6 - 1 = 0 \pmod 7$ by $(*)$

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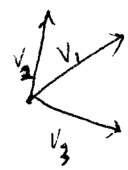
[A3]. * P.27 : 6 points are impt.

Linear combinations in $\mathbb{R}^3$



linear combo of v_1 & v_2 & v_3 are vectors of form $av_1 + bv_2 + cv_3$

Span $\{v_1, v_2\} := \{$all linear combos of v_1 & $v_2\}$.
= plane containing v_1 & v_2 (in $\mathbb{R}^3$)

Span of any collection of vectors is a subspace

eg. Span of v_1 is line
Span of v_1 & v_2 is plane.

eg. $(-2,0,3), (1,3,0), (2,4,-1)$ Write 2 vectors as linear combo of other 2..

$a(-2,0,3) + b(1,3,0) + c(2,4,-1) = 0.$

$$\begin{cases} -2a + b + 2c = 0. \\ 3b + 4c = 0 \\ 3a - c = 0 \end{cases} = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 4 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve, by using ops in book.