Read Along : BDP Ch. 1, 2.1~2.2, 2.4~2.6

"Differential Equation"

"Ordinary" where function is a function of just 1-variable.

"Partial" functions of many variables. Equations involve $f_{xy}, \dots$

"order" $y' = y$ order 1

$y'' = -y$ order 2

Type 0

$y' = f(x)$

Solution

$$y = \int f(x)$$

$$= F(x) + c$$

Type 1 "1st order linear homogeneous"

$a(x)\mu' + b(x)\mu = 0$

$L : \{\text{functions on } \mathbb{R}\} \longrightarrow \{\text{functions on } \mathbb{R}\}$

$L\mu = a\mu' + b\mu$

linear operator

$L(\mu_1 + \mu_2) = L(\mu_1) + L(\mu_2)$

Equation : $L\mu = 0$

$\Longleftrightarrow \mu' = P(x)\mu \quad P = -\frac{b}{a}$

$\Longleftrightarrow \frac{\mu'}{\mu} = P(x)$

$\Longleftrightarrow (\log|\mu|)' = P(x)$

$\log|\mu| = \int P(x)dx + c$

$\Longrightarrow |\mu| = e^{\int P(x)} \underbrace{c_1}_{>0} \quad P = -\frac{b}{a}$

$\Updownarrow$

$\mu = e^{\int P(x)} \cdot c_2$. c_2 is arbitrary

<u>example</u>

$$t\dot{\mu} = 2\mu$$

$$\frac{\dot{\mu}}{\mu} = \frac{2}{t}$$

$$\boxed{\ddot{\mu} = \frac{d^2\mu}{dt^2}}$$

$$(\log|\mu|)' = \frac{2}{t}$$

$$\log|\mu| = 2\log|t| + c$$

$$|\mu| = c_1 \cdot e^{2\log|t|}$$

$$= c_1(e^{\log|t|})^2$$

$$= c_1|t|^2$$

$$= c_1 t^2$$

$$\Rightarrow \mu = ct^2, \quad c \in \mathbb{R}$$

<u>True?</u>

$$t \cdot (ct^2)' \stackrel{?}{=} 2(ct^2) \quad \checkmark$$

<u>Type 2 "1st order linear, non-homogeneous"</u>

$$a(x)y' + b(x)y = c(x), \qquad L\mu = c$$

$$\Leftrightarrow y' + p(x)y = g(x) \qquad p = \frac{b}{a}, \quad g = \frac{c}{a}$$

<u>Trick</u>: Multiply equation by an "integrating factor" μ, so that the LHS would be a derivative

$$\underline{\text{RHS}} \qquad \underline{\text{LHS}}$$

$$g\mu = \mu y' + p\mu y = \underbrace{(\mu \cdot y)'}_{\mu y' + \mu' y}$$

(μ must satisfy $\mu' = p\mu$: $\mu = e^{\int p} \cdot \cancel{c_1}$)

So $\int g\mu\, dx = \mu y + c_2$

$$y = -\frac{c_2}{\mu} + \frac{1}{\mu}\int g\mu\,dx$$

$$= \frac{c}{e^{\int p}} + \frac{1}{e^{\int p}}\int g e^{\int p} dx$$

example

Solve $ty' + 2y = 4t^2 \quad y(1) = 2$

Solution

Wish the following

$t\mu y' + 2\mu y = ((t\mu)\cdot y)'$

$$2\mu = (t\mu)' = t\mu' + \mu$$

$$t\mu' = \mu$$

$$\frac{\mu'}{\mu} = \frac{1}{t}$$

$$\log|\mu| = \int \frac{1}{t} = \log|t|$$

$$|\mu| = |t|$$

Take $\mu = t$, equation becomes

$$(t\mu\cdot y)' = 4t^2\mu$$

$$(t^2 y)' = 4t^3$$

$$t^2 y = t^4 + c$$

$$y = t^2 + \frac{c}{t^2}$$

$$2 = y(1) = 1 + \frac{c}{1} \Rightarrow c = 1$$

$$\therefore y = t^2 + \frac{1}{t^2}$$

Solves $y' = 4t - \frac{2y}{t}$