

hw #1

$$\begin{aligned}
 & 1 \quad a^{-1}b^{-1} \stackrel{0}{=} (ab)^{-1} \\
 & = (ab)(a^{-1}b^{-1}) \\
 & = (ba)(a^{-1}b^{-1}) \quad (\text{Commutativity}) \\
 & = b \cdot (a(a^{-1}b^{-1})) \quad (\text{Associativity}) \\
 & = b \cdot ((aa^{-1})b^{-1}) \quad (\text{Assoc.}) \\
 & = b \cdot (1 \cdot b^{-1}) \quad (\cdot \text{inverse}) \\
 & = b \cdot b^{-1} \quad (\cdot \text{identity}) \\
 & = 1 \quad (\cdot \text{inverse})
 \end{aligned}$$

3.1  $F_1 = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$   $+$ ,  $\cdot$  are addition & mult of real #s

(1) Commutative, associative, distributive law are true by ~~real numbers~~  
~~So don't need to show~~ since they are true for any real numbers.

(2) additive identity =  $0 + 0\sqrt{3} = 0$   
 mult identity =  $1 + 0\sqrt{3} = 1$

(3) additive inverse of  $a + b\sqrt{3} = -a - b\sqrt{3}$

multiplicative inverse  ~~$(a + b\sqrt{3})^{-1} = c + d\sqrt{3}$~~

Assume  $(a + b\sqrt{3})^{-1} = c + d\sqrt{3}$

Find  $c$  &  $d$  in terms of  $a$  &  $b$

$$\begin{aligned}
 (a + b\sqrt{3})(c + d\sqrt{3}) &= 1 \\
 ac + 3bd + bc\sqrt{3} + ad\sqrt{3} &= 1 \\
 (ac + 3bd) + (bc + ad)\sqrt{3} &= 1 + 0\sqrt{3} \\
 ac + 3bd = 1 \quad (1) &, \quad bc + ad = 0 \\
 & \quad bc = -ad \quad (2)
 \end{aligned}$$

If  $a \neq 0, b = 0$

$$-\frac{bc}{a} = d \quad (3)$$

Sub (3) into (1)

$$ac - \frac{3b^2c}{a} = 1$$

$$\frac{c(a^2 - 3b^2)}{a} = 1$$

$$c = \frac{a}{a^2 - 3b^2}$$

$$d = \frac{-b}{a^2 - 3b^2}$$

For  $a = 0, b \neq 0$

Assume  $a + b\sqrt{3} \neq 0$

$$\begin{aligned}
 \Leftrightarrow a - b\sqrt{3} &\neq 0 \\
 (a + b\sqrt{3})(a - b\sqrt{3}) & \\
 &= a^2 - 3b^2
 \end{aligned}$$



# MAT 240 - Tutorial

w/  $a+1=b$

$$a+1=0 \quad \not\Rightarrow a=1$$

$$\Leftrightarrow a+(1+1)=0+1=1$$

$$\Leftrightarrow a=1 \quad \not\Leftarrow$$

$$a+1=1$$

$$\Leftrightarrow a=0 \quad \rightarrow \Leftarrow$$

$$a+1=0$$

$$\Leftrightarrow 1=0 \quad \rightarrow \Leftarrow$$

## eg of v.s.

•  $\mathbb{C}$  is a v.s. over  $\mathbb{R}$ .

•  $\{0\}$  over any field

• matrices over any field ie.  $M_{m \times n} = \{m \times n \text{ matrices over any field}\}$   
↳ of some dimension

•  $\mathbb{R}^n$  is a v.s.,  $n \in \{0, 1, 2, \dots\}$

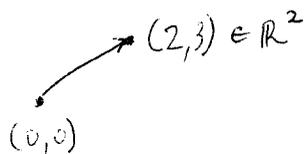
$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}$$

↳ tuple of  $n$

↳  $i \times n$  matrix

eg.  $(1.2, 4.53) \in \mathbb{R}^2$

$\mathbb{R}^2$  is a plane



$$\begin{aligned}
 P(F) &= \{\text{polynomials over a field } F\} \\
 &= \{a_0 + a_1x + a_2x^2 + \dots + a_mx^m \mid a_i \in F\}
 \end{aligned}$$

~~Let~~

Let  $p$  be a polynomial over  $F$

$$p = a_0 + a_1x + \dots + a_mx^m \text{ for some } F.$$

The smallest  $n$  s.t.  $a_n \neq 0$  is called the degree of  $f$

eg. degree of  $x^3 + 3x^2 + 1 = 3$ .

$$P_n(F) = \{\text{polynomials of degree } \leq n\}$$

$$P_2(\mathbb{R}) = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbb{R}^3 & & & \\ & (1,0,0) & (0,1,0) & (0,0,1) \end{array}$$

eg  $V = \{(a_1, a_2) \in \mathbb{R}^2 \mid 2a_1 + a_2 = 0\}$  is a vector space over  $\mathbb{R}$ .  
 (addition & scalar multiplication inherit from  $\mathbb{R}^2$ )

(V1) commutativity of  $+$   $v_1 + v_2 = v_2 + v_1$

(V2) associativity of  $+$   $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$

$\hookrightarrow$  both follow from fact that  $\mathbb{R}^2$  is a vect

(V3) existence of  $0$   $0 = (0,0)$   
 $(0,0) \in V$   
 $\therefore 2(0) + 0 = 0$ .

Before all these checks, we should check that  $+$  &  $\cdot$  are operations on  $V$ .

Check: Let  $(a_1, a_2) \in V, (b_1, b_2) \in V$   
 $(a_1, a_2) + (b_1, b_2) \in V$   
 $(a_1 + b_1, a_2 + b_2)$

Check that  $2(a_1 + b_1) + (a_2 + b_2) = 0$  provided that  $2a_1 + a_2 = 0, 2b_1 + b_2 = 0$

## MAT 240 - Tutorial

$$\begin{aligned}
 & 2(a_1 + b_1) + a_2 + b_2 \\
 &= (2a_1 + a_2) + (2b_1 + b_2) \\
 & \text{in } (0,0) + (0,0) \\
 &= 0 + 0 = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{let } r \in \mathbb{R}, r(a_1, a_2) & \stackrel{?}{\in} V \\
 & \text{''} \\
 & (ra_1, ra_2)
 \end{aligned}$$

Need to check that

$$\begin{aligned}
 & 2(ra_1) + ra_2 = 0 \\
 \text{provided } & \text{2}a_1 + a_2 = 0
 \end{aligned}$$

$$2ra_1 + ra_2 = r(2a_1 + a_2) = 0$$

3) Existence of 0. Need to check  $0 + v = v \quad \forall v \in V$

$$\text{let } (a_1, a_2) \in V \Leftrightarrow 2a_1 + a_2 = 0$$

$$\begin{aligned}
 & (0, 0) + (a_1, a_2) \\
 &= (0 + a_1, 0 + a_2) = (a_1, a_2)
 \end{aligned}$$

4) Existence of negative

Given any  $x \in V$ , there exists  $y \in V$  s.t.  $x + y = 0$

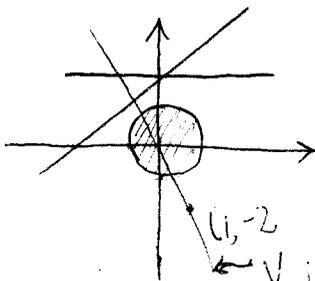
$$\text{let } x = (a_1, a_2) \in V$$

$$\text{then } y = (-a_1, -a_2) \in V$$

↑ check this as exercise

$$V = \{(a_1, a_2) \in \mathbb{R}^2 \mid 2a_1 + a_2 = 0\}$$

not vector spaces



← V is a vector space

\* What are the vector spaces in  $\mathbb{R}^2$ ?  
 $\{(0,0)\}$ ,  $\mathbb{R}^2$  <sup>plane</sup>, any straight line that passes through  $(0,0)$   
Exercise: Prove this after next week.

eg.  $C([0,1]) = \{ \text{continuous real valued functions on } [0,1] \}$  over  $\mathbb{R}$   
 $\rightarrow \infty$  dimensional vector space

$$f_1, f_2 \in C([0,1])$$

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$\text{If } r \in \mathbb{R}, (rf_1)(x) = r(f_1(x))$$

$$P(\mathbb{R}) = \{ \text{polynomials over } \mathbb{R} \} \in C([0,1])$$

$$\bigcup P_n(\mathbb{R})$$

$\rightarrow$  for each  $n$  there is a vector space in  $P(\mathbb{R})$ , therefore  $\infty$  it: