$$= \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \partial f_m/\partial x_1 & \vdots & & \partial f_m/\partial x_n \end{pmatrix}$$ "Jacobian matrix of f"

* $f(x,y) = \begin{pmatrix} e^x \cos y \\ e^y \cos x \\ \frac{1}{1+x^2+y^2} \end{pmatrix}$ $\mathbb{R}^2 \Rightarrow \mathbb{R}^3$ $\quad \underset{3\times 2}{Df} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \\ \frac{-2x}{(1+x^2+y^2)^2} & \frac{-2y}{(1+x^2+y^2)^2} \end{pmatrix}$

Pf. (of thm 1). Assume $f(a+h) = f(a) + B_1 h + o(h)$

$f(a+h) = f(a) + B_2 h + o(h)$.

$(B_1 - B_2)h + (\phi_1 - \phi_2)$ $\quad \phi_1, \phi_2 \in o(h)$. $o(h)$ is a v.s.

$\Rightarrow (B_1 - B_2)h \in o(h)$, i.e. $(B_1 - B_2)h/|h| \to 0$.

~~th then~~ $B_1 - B_2 = 0$. $\therefore B_1 = B_2$ $\square$.

(of thm 2). $f(x) = C \in \mathbb{R}^m$ $\forall x$ $\quad f(x+h) = f(x) + Bh + o(h)$.

$C = C + 0h + 0 \Rightarrow (Df)(a) = 0$.

(of thm3) $f(x) = Ax$. $f(a+h) = A(a+h) = A(a) + Ah = f(a) + Ah + 0$. $Df(a) = A$.

(of thm4) $(f+g)(a+h)$ ...

(of thm5) $f'(a;u) = \lim_{h\to 0} \frac{f(a+h\cdot u) - f(a)}{h} = \lim_{h\to 0} \frac{f(a) + Df(a)\cdot h \cdot u + o(h\cdot u) - f(a)}{h} = Df(a).u$.

Oct 12.

$f(a+h) = f(a) + Df(a)\cdot h + o(h)$. $\iff \lim_{h\to 0} \frac{f(a+h) - f(a) - Df(a)\cdot h}{|h|} = 0$.

If f is diff then $Df(a) = \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ & & \vdots \\ \partial f_m/\partial x_1 & \cdots & \partial f_m/\partial x_n \end{pmatrix}$ "Jacobian matrix"

$f: \mathbb{R}^n \to \mathbb{R}^m$

* Check: $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff $\iff \forall k$, $f: \mathbb{R}^n \to \mathbb{R}$ is diff.

Thm: $f: \mathbb{R}^n \to \mathbb{R}$. assume $\frac{\partial f}{\partial x_i}$ exist and are cont. near a_n. Then f is diff. at a.

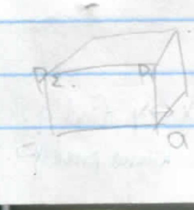
Lemma: For any small $h \in \mathbb{R}^n$ $\exists q_1, \dots q_n$ in $U(a, |h|)$ s.t. $f(a+h) - f(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(q_i)\cdot h$

If $g: \mathbb{R} \to \mathbb{R}$ s.t. f' exists & is cont. near a, and h is "small" then $f(a+h) - f(a) = f'(q)\cdot h$

(the mean value theorem, proved in 157) $\iff \frac{f(a+h) - f(a)}{h} = f'(q)$. h small. like tangent.

Pf. (lemma $\Rightarrow$ thm): $B = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right)$

~~f(a+h)~~ $\frac{f(a+h) - f(a) - Bh}{|h|} = \frac{\sum \frac{\partial f}{\partial x_i}(q_i)h_i - \sum \frac{\partial f}{\partial x_i}(a)h_i}{|h|} = \frac{\sum \left(\frac{\partial f}{\partial x_i}(q_i) - \frac{\partial f}{\partial x_i}(a)\right)h_i}{|h|} \xrightarrow{h\to 0} 0$.



Pf. (lemma) Let $p_0 = a$. $p_1 = a + h_1 \cdot e_1$ $\quad p_2 = p_1 + h_2 \cdot e_2$, $\cdots$ $p_n = p_{n-1} + h_n e_n = a + h_1 e_1 + \cdots + h_n e_n$.

telescopic summation

Then, $f(a+h)-f(a)=\sum_{i=1}^{n} F(P_i)-F(P_{i-1}) = \sum_{i=1}^{n}\left(f(P_{i-1}+h_ie_i)-f(P_{i-1})\right)$

(by MVT) $\Rightarrow \exists t_i\in[0,1]$ s.t. $q_i = P_{i-1}+t_ih_ie_i$ then $\sum_{i=1}^{n} \partial f/\partial x_i\,(q_i)\cdot h_i$ $\square$

Def. f is of class C^1 on $A\subset \mathbb{R}^n$ if $\partial f/\partial x_i$ exist and are cont. on A.

Def. f is of class C^r $(r\in\mathbb{N}, r\geq 1)$ if $\partial f/\partial x_i$ exist and are of class C^{r-1}

Def. A function is C^0 ~~is~~ if it is cont.

* $f(x,y)$ is C^2 iff $\partial_x f$, $\partial_y f$ are C^1

iff $\partial_x\partial_x f$, $\partial_x\partial_y f$, $\partial_y\partial_x f$, $\partial_y\partial_y f$ are cont.

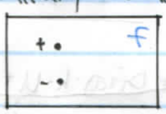
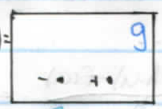
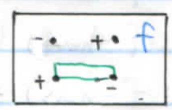
Thm. If $f:\mathbb{R}^n\to\mathbb{R}$ is C^2, then for any i,j, $\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)=\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$ $\partial_i\partial_j f=\partial_j\partial_i f$

Oct 14

e.g. $x^2y \xrightarrow{\partial x} 2xy \xrightarrow{\partial y} 2x$

$x^2y \xrightarrow{\partial y} x^2 \xrightarrow{\partial x} 2x$

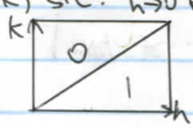
* WLOG $f:\mathbb{R}^2_{x,y}\to\mathbb{R}$

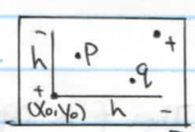
~~∂f~~ $\partial_y f$ $\overset{\shortparallel}{g}$:  $\partial_x(\partial_y f)$ =  $\partial_x(\partial_y f)$ =  $=\partial_y(\partial_x f)$.

topology mapping from $\mathbb{R}^2\to\mathbb{R}$

the problem is: diff. also take the limit, not just difference.

e.g. Can we find $f(h,k)$ s.t. $\lim_{h\to 0}\lim_{k\to 0} f(h,k)\neq \lim_{k\to 0}\lim_{h\to 0} f(h,k)$

e.g. $f(h,k)=\begin{cases}0 & h\leq k\\ 1 & h>k\end{cases}$  then $\lim_{h\to 0}\lim_{k\to 0} f(h,k)=\lim_{h\to 0}(1)$; $\lim_{k\to 0}\lim_{h\to 0} f(h,k)=\lim_{k\to 0}(1)$.

proof.  Lemma: $\exists\, p,q\in[x_0,x_0+h]\times[y_0,y_0+h]$

$$\partial_x\partial_y F(p)=\frac{F(x_0,y_0)-F(x_0+h,y_0)-F(x_0,y_0+h)+F(x_0+h,y_0+h)}{h^2}=(\partial_y\partial_x F)(q)$$

$p\to\binom{x_0}{y_0}$ $\quad\downarrow h\to 0\quad$ $q\to\binom{x_0}{y_0}$

$$(\partial_x\partial_y F)\binom{x_0}{y_0} \;\overset{?}{=}\; (\partial_y\partial_x F)\binom{x_0}{y_0}$$

* If f is cont. $\lim_{h\to 0}\lim_{k\to 0}\varphi(h,k)=\lim_{k\to 0}\lim_{h\to 0}\varphi(h,k)$ $\quad\varphi(h,k)=\frac{f(x_0,y_0)-f(x_0+h,y_0)-f(x_0,y_0+h)+\cdots}{hk}$

let $g(x)=\frac{f(x,y_0+h)-f(x,y_0)}{h}$ $\quad\lambda=\frac{g(x_0+h)-g(x_0)}{h}$

By MVT $\exists x_1\in[x_0,x_0+h]$ s.t. $\lambda=g'(x_1)=\frac{\overbrace{\partial_x F(x_1,y_0+h)-\partial_x F(x_1,y_0)}^{z}}{h}$

$z(y)=\partial_x F(x_1,y)$ $\quad = \partial_y z(y_1)=\partial_y\partial_x F(x_1,y_1)$ set $p=\binom{x_1}{y_1}$. Do same with $x\leftrightarrow y$ find q.

by MVT for z $\exists y_1$

(lemma proved) $\square$.