Def: Let V be a V.S. over F. let $\{u_1, \dots u_n\}$ be vectors in V, a vector $u \in V$ is a linear combination of $\{u_1, \dots, u_n\}$ if ~~there~~

$$\exists \alpha_1, \dots \alpha_n \in F \text{ s.t. } u = \alpha_1 u_1 + \dots \alpha_n u_n$$
$$= \sum_{i=1}^{n} \alpha_i u_i$$

If $S \subseteq V$ the "span of S"

$$\text{Span}(S) = \{ u \in V : u \text{ is a linear combination of } \{u_1, \dots, u_n\} \subset S \}$$

ex: No matter what S is $0 \in \text{Span } S$ even if $S = \emptyset$ {ie can take all α to be 0}

ex In $P(\mathbb{R})$

$u = 2x^3 - 2x^2 + 12x - 6$ is a linear combination of $u_1 = x^3 - 2x^2 - 5x - 3$ & $u_2 = 3x^3 - 5x^2 - 4x - 9$

$\exists ? \ \alpha_1, \alpha_2$ st. $u = \alpha_1 u_1 + \alpha_2 u_2$

$$\alpha_1 u_1 + \alpha_2 u_2 = \alpha_1 (x^3 - 2x^2 - 5x - 3) + \alpha_2 (3x^3 - 5x^2 - 4x - 9)$$
$$= (\alpha_1 + 3\alpha_2)x^3 + (-2\alpha_1 - 5\alpha_2)x^2 + (-5\alpha_1 - 4\alpha_2)x + (-3\alpha_1 - 9\alpha_2)$$
$$\stackrel{?}{=} 2x^3 - 2x^2 + 12x - 6$$

$\Longleftarrow$ $\alpha_1 + 3\alpha_2 = 2$ $\quad -5\alpha_1 - 4\alpha_2 = 12$

$-2\alpha_1 - 5\alpha_2 = -2$ $\quad -3\alpha_1 - 9\alpha_2 = -6$

$6\alpha_2 - 5\alpha_2 = 2 \qquad \alpha_2 = 2$

$\Rightarrow \alpha_1 + 3\cdot 2 = 2 \quad \Rightarrow \alpha_1 = -4$

$\cancel{\alpha = 2} \quad \alpha_1 = -4, \ \alpha_2 = 2.$

$\Rightarrow$ indeed so u is a linear combination of u_1 and u_2

<u>Theorem:</u> Given $S \underset{\text{subset of v.s.}}{\subseteq} \underset{\text{v.s.}}{V}$ span S is a subspace of V.

<u>Proof:</u> $W := \text{span}(S)$

0. $0 \in W$ because you can always take empty linear comb.

1. Suppose $x \in W \Rightarrow$ For some $u_1, \dots, u_n \in S$. $x = \sum \alpha_i u_i$

   $y \in W \Rightarrow$ For some $v_1, \dots, v_m \in S$ $y = \sum \beta_j \alpha v_j$

   (S could be infinite set so can't enumerate all element)

$$x + y = \sum_{i=1}^{n} \alpha_i u_i + \sum_{j=1}^{m} \beta_j v_j$$

$= \alpha_1 u_1 + \dots \alpha_n u_n + \beta_1 v_1 + \dots + \beta_n v_m$, but this is a linear combination of $u_1, \dots u_n, v_1, \dots v_m \in S$

So $x + y \in \text{span}\, S.$ ✓

2. W is closed under multiplication by a scalar