

Homework 8. NOT HARD IN.

Section 12

1. prove $U(f, P) \geq U(\bar{I}, P_A)$ where $\bar{I} = \int_{y \in B} f(x, y)$

proof: let x_0 be an arbitrary vector in A . s.t. $x_0 \in P_A$,

& $P_A \times P_B$ is a subrectangle of determined by the partition of $P \Rightarrow$

then $M_{P_A \times P_B} f(x, y) \geq M_{P_A} f(x_0, y)$ given $x_0 \in P_A$ since $M_{P_A \times P_B} f \geq f(x, y)$

Multiply both sides by $V(P_B)$ & sum it $\sum_{y \in B} V(P_B) f(x_0, y) \geq \sum_{y \in B} V(P_B) M_{P_A} f(x_0, y)$ for all $y \in B$

we get $\sum_{y \in B} V(P_B) M_{P_A \times P_B} f(x, y) \geq \sum_{y \in B} V(P_B) M_{P_A} f(x_0, y) = U(f(x_0, \cdot), P_B) \geq \int_{y \in B} f(x_0, y)$

since x_0 is arbitrary, so it is applied to all x_0

then multiply both sides by $V(P_A)$ & sum it

we get $\sum_{x \in P_A} \sum_{y \in B} V(P_A) V(P_B) M_{P_A \times P_B} f(x, y) \geq \sum_{x \in P_A} V(P_A) \int_{y \in B} f(x, y)$

$\Rightarrow U(f, P) \geq U(\bar{I}, P_A)$ as required

since this result holds for each $x_0 \in P_A$, then $\sum_{y \in B} M_{P_A \times P_B} f(x, y) V(P_B) \geq M_{P_A}(\bar{I})$

2. $Q = [0, 1]^2$ $f: Q \rightarrow \mathbb{R}$ $f(x, y) = \begin{cases} \frac{1}{q} & x = \frac{p}{q}, y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

a) given $\epsilon > 0$, let $\{q_i\}$ be a list of rational # in $[0, 1]$ & $\{q_j\}$ as well. the function is discont at $x \in \mathbb{Q}$ & $y \in \mathbb{Q}$.

let P_1 be a partition of $[0, 1]$ determined by $[q_i - \frac{0.9\epsilon}{2^{1/n}}, q_i + \frac{0.9\epsilon}{2^{1/n}}]$

let P_2 be a partition of $[0, 1]$ determined by $[q_j - \frac{0.9\epsilon}{2^{1/n}}, q_j + \frac{0.9\epsilon}{2^{1/n}}]$

so $P = \{P_1, P_2\}$ is a partition of $[0, 1]^2$ with subrectangles R s.t. VR covers $\{x, y\}$ $x \in \mathbb{Q}, y \in \mathbb{Q}$

s.t. $\sum_{R \in P} V(R) = \sum_{i=1}^n \sum_{j=1}^n \frac{0.9\epsilon}{2^{1/n}} \cdot \frac{0.9\epsilon}{2^{1/n}} = \sum_{i=1}^n \frac{0.9\epsilon}{2^{1/n}} \cdot 0.9\epsilon = (0.9\epsilon)^2 = 0.81\epsilon < \epsilon$

So the discont. set $\{(x, y) : x \in \mathbb{Q}, y \in \mathbb{Q}\}$ has measure 0 on Q .

$\Rightarrow \int_Q f$ is integrable by TMM.

b) compute $\int_{y \in B} f(x, y)$

if $x_0 = \frac{p}{q}$ then $L(f(x_0, \cdot), P_B) = \sum_{R \in P_B} V(R) m_R f(x_0, y) = 0$ for an arbitrary partition P_B since \exists irrational # in any interval of $[0, 1]$ so $m_R f(x_0, y) = 0$.

so $\int_{y \in B} f(x_0, y) = \sup_{P_B} L(f(x_0, \cdot), P_B) = 0$.

if $x_0 \neq \frac{p}{q}$ then $f(x_0, y) = 0$ & $L(f(x_0, \cdot), P_B) = 0$ for an arbitrary partition P_B

so $\int_{y \in B} f(x_0, y) = \sup_{P_B} L(f(x_0, \cdot), P_B) = 0$.

Hence whatever x is, $\int_{y \in B} f(x, y) = 0$.

c) compute $\int_{y \in B} f(x, y)$

if $x_0 = \frac{p}{q}$ then $U(f(x_0, \cdot), P_B) = \sum_{R \in P_B} V(R) M_R f(x_0, y) = \sum_{y \in B} V(P_B) \frac{1}{q} = \frac{1}{q}$ since $\exists y \in \mathbb{Q}$ in every interval.

so $\int_{y \in B} f(x_0, y) = \inf_{P_B} U(f(x_0, \cdot), P_B) = \frac{1}{q}$

if $x_0 \neq \frac{p}{q}$, then $U(f(x_0, \cdot), P_B) = 0$ since $f(x_0, y) = 0$ for all y for an arbitrary partition P_B .

$\int_{y \in \mathbb{R}} f(x, y) dy = 0$ $\int_{x \in \mathbb{R}} f(x, y) dx = 0$
 Hence, ~~whether x is~~ $\int_{y \in \mathbb{R}} f(x, y) dy = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \\ 0 & x \neq \frac{p}{q} \end{cases}$
 c) since $\int_{y \in \mathbb{R}} f(x, y) dy = 0$ so it is integrable. & $\int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} f(x, y) dx dy = 0$
 since $\int_{y \in \mathbb{R}} f(x, y) dy = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \\ 0 & \text{otherwise} \end{cases}$ it is cont when $x \neq \frac{p}{q}$, i.e. $x \notin \mathbb{Q}$.

let $\frac{1}{n} > 0$ be given, $\int_{y \in \mathbb{R}} f(x, y) dy = g(x, y) \geq \frac{1}{n}$ when it takes value $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$.
 $f(x, y) \geq \frac{1}{n}$ happens for at most $n!$ (finite many) in $[0, 1]^2$.
 so we can pick $\delta > 0$ s.t. $\forall x \in B_\delta(x_0)$ $g(x, y) < \frac{1}{n}$. $S = \inf(d(x, x_0), d(x_0, x_1) - \dots)$
 we are shown that g is cont on $(\mathbb{Q}^c \cap [0, 1]) \times [0, 1]$

so g is discont. in $(\mathbb{Q} \cap [0, 1]) \times [0, 1]$ which has measur. = 0.

so g is integrable, & since g is 0 almost everywhere.

$$\text{so } \int_{\mathbb{R}^2} g = \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} f(x, y) dx dy = 0$$

Hence both $\int_{y \in \mathbb{R}} f(x, y) dy$ & $\int_{x \in \mathbb{R}} f(x, y) dx$ is integrable over $[0, 1]$

$$\& \int_{\mathbb{R}^2} f = \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} f(x, y) dx dy = \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} f(x, y) dy dx = 0 \quad \text{verified Fubini TMM}$$

4 a) let $Q = [a, b] \times [c, d]$. The function $(x, y) \mapsto D_1 f(x, y)$ & $(x, y) \mapsto D_2 f(x, y)$ are cont. on Q since $f \in C^2$.
 By Fubini TMM, $\int_Q D_2 f(x, y) dy dx = \int_a^b \left[\int_c^d D_2 f(x, y) dy \right] dx$
 By the fundamental TMM of calculus, $\int_c^d D_2 f(x, y) dy = D_1 f(x, d) - D_1 f(x, c)$
 so $\int_Q D_2 f(x, y) dy dx = \int_a^b [D_1 f(x, d) - D_1 f(x, c)] dx = f(b, d) - f(a, d) - f(b, c) + f(a, c)$

Applying Fubini's TMM to $D_1 f$.

$$\text{get } \int_Q D_1 D_2 f(x, y) dx dy = \int_c^d \left[\int_a^b D_1 D_2 f(x, y) dx \right] dy = f(b, d) - f(a, d) - f(b, c) + f(a, c)$$

$$\text{Hence } \int_Q D_2 D_1 f = \int_Q D_1 D_2 f$$

b) let $E = \{(x, y) \in A \mid D_2 D_1 f(x, y) > D_1 D_2 f(x, y)\}$. Since $D_2 D_1 f$ & $D_1 D_2 f$ are cont.

the set E is open.

if $E \neq \emptyset$, then \exists rectangle $Q \subset E$. then $\int_Q (D_2 D_1 f - D_1 D_2 f) > 0$. \Rightarrow part a

$$\text{so } E = \emptyset$$

similarly the set $E' = \{(x, y) \in A \mid D_1 D_2 f(x, y) > D_2 D_1 f(x, y)\} = \emptyset$.

Hence $D_2 D_1 f = D_1 D_2 f$ at every point of A .