November 24th, 2009

MAT240: Abstract Linear Algebra Lecture:

Apparently, the next class will be awful.

In general, variables $x_1 \dots x_n$ $a_{ij} \in F$ $b_1 \dots b_m \in F$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, (unknown vector), $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in F^m$

$\rightarrow Ax = b$

Solve for the coordinates of unknown vector.

If $b = 0 \rightarrow Ax = 0 \quad \text{homogeneous}$

If $b \neq 0 \rightarrow Ax = b \quad \text{inhomogeneous}$

If lucky, A is invertible, i.e, there is $A^{-1}\ s.t. A^{-1}A = I$

$$Ax = b \rightarrow A^{-1}A = A^{-1}b \rightarrow x = A^{-1}b$$

So equation has a unique solution and it is $x = A^{-1}b$

Observations regarding the homogeneous case ($Ax = 0$):

1. $Ax = 0 \leftrightarrow T_A(x) = 0$
   $$x \in nullspace(T_A) = N(T_A) = Ker(T_A) = N(A) = Ker(A)$$
2. The set of solutions in $N(A)$, is a subspace of F^n.
   $$dimension(solution) = nullity(A)$$
3. 0 is always a solution.

Observations regarding the inhomogeneous case ($Ax = b$):

1. Solutions exist iff $b \in R(A) = image(A) = column - span(A)$
2. If x_0 is a solution of ($Ax = b$), then x_1 is a solution too if $x_1 = x_0 + x$ where x is a solution of $Ax = 0$.
   Proof:

   Write $x_1 = x_0 + x$

   x_1 is a solution $\leftrightarrow Ax_1 = b \leftrightarrow A(x_0 - x) = b \leftrightarrow Ax_0 + Ax = b \leftrightarrow b + Ax = b \leftrightarrow Ax = 0$

Moral: To find all solutions of $Ax = b$, find one solution and add to it all solutions of $Ax = 0$

Consider geometric examples in R^3. For $Ax = 0$ consider the solution to be a plane passing through the origin. For $Ax = b$ consider the solution to be a plane travelling through b.

$$Ax = b \leftrightarrow E_1Ax = E_1b \leftrightarrow E_2E_1Ax = E_2E_1b \leftrightarrow Cx = d$$

Where C is row-reduced and d is the result of applying same operations to b.

$(A|b) \xrightarrow{row\ reduce} (C|d)$

Example: solve Cx=d where:

$$C = \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

Note: the above matrix A represents the system of linear equations:

$x_1 + 2x_3 - 2x_5 = d_1$

$x_2 - x_3 + x_5 = d_2$

$x_4 - 2x_5 = d_3$

$0 = d_4$

If $d_4 \neq 0$, no solutions assume $d_4 = 0$.

The following is a general solution to $Cx = 0$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_5 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Finally:

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

The pivotal columns contain a "pivot", that is to say a leading 1. Non-pivotal columns do not contain a pivot. In this matrix, the pivotal variables (the variables corresponding to a pivotal column) are x_1, x_2, x_4. The free variables (the variables not corresponding to a pivotal column) are x_3, x_5. In addition, d_1, d_2, d_3 are pivotal rows, while d_4 is a non-pivotal row.

1. A solution exists only if the d_i's in the non-pivotal rows are all 0.
2. In that case, the "free" variables corresponding to non-pivotal columns can be set arbitrarily, and other variables are fixed by the equations.

Eg.

$$2x_1 + 2x_2 + x_3 + 4x_4 - 9x_5 = 17$$

$$x_1 + x_2 + x_3 + x_4 - 3x_5 = 6$$

$$x_1 + x_2 + x_3 + 2x_4 - 5x_5 = 8$$

$$2x_1 + 2x_2 + 2x_3 + 3x_4 - 8x_5 = 14$$

$$\rightarrow \left(\begin{array}{ccccc|c} 2 & 2 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right) \xrightarrow{row\ reduction} \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ the general solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_5 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$