

Problem 3 Proof:

① " \Leftarrow " when $W_1 \subset W_2$ or $W_2 \subset W_1$,
we have $W_1 \cup W_2 = W_2$ or $W_1 \cup W_2 = W_1$
 $\therefore W_1, W_2$ are all subspaces of V
 $\therefore W_1 \cup W_2$ must be a subspace of V

② " \Rightarrow " we just suppose that the statement is wrong
which means $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$
Then $\exists x \in W_1$ s.t. $x \notin W_2$ & $\exists y \in W_2$ s.t. $y \notin W_1$
 \Leftrightarrow Then $x \in W_1 \cup W_2$, $y \in W_1 \cup W_2$
 $\therefore W_1 \cup W_2$ is a subspace of V
 $\therefore x+y \in W_1 \cup W_2$

(i) if $x+y \in W_1 \cap W_2$, then $x+y \in W_1$ & $x+y \in W_2$
In W_1 , $x \in W_1$, $x+y \in W_1 \Rightarrow x+y - x \in W_1$,
which means $y \in W_1$, there is a contradiction with
 $y \notin W_1$

(ii) if $x+y \in W_1$ and $x+y \notin W_2$, then
In W_1 , $x \in W_1$, $x+y \in W_1 \Rightarrow x+y - x \in W_1$,
which means $y \in W_1$, there is a contradiction with
 $y \notin W_1$

(iii) if $x+y \in W_2$ and $x+y \notin W_1$, then 25
In W_2 , $x \in W_2$, $x+y \in W_2 \Rightarrow x+y - x \in W_2$ Well done!
which means $y \in W_2$, there is a contradiction with $y \notin W_2$

Hence $W_1 \subset W_2$ or $W_2 \subset W_1$
 $\Rightarrow W_1 \cup W_2$ is a subspace of V iff $W_1 \subset W_2$ or $W_2 \subset W_1$

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Problem 4. Solution : Yes

We suppose that P is a linear combination of the polynomials $U_1 = X^3 - X$, $U_2 = X^2 + 1$, $U_3 = X^3 + X^2$
Then $\exists a_1, a_2, a_3 \in \mathbb{Q}$ s.t.

$$\begin{aligned} P &= a_1 U_1 + a_2 U_2 + a_3 U_3 \\ &= a_1(X^3 - X) + a_2(X^2 + 1) + a_3(X^3 + X^2) \\ &= (a_1 + a_3)X^3 + (a_2 + a_3)X^2 - a_1X + a_2 \end{aligned}$$

Then we can get a system of linear equations:

$$\begin{cases} a_1 + a_3 = 1 & \textcircled{1} \\ a_2 + a_3 = 2 & \textcircled{2} \\ -a_1 - a_2 = -3 & \textcircled{3} \\ a_2 = 4 & \textcircled{4} \end{cases}$$

exchange $\textcircled{1}$ & $\textcircled{3}$, $\textcircled{2}$ & $\textcircled{4}$

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 $\Rightarrow \begin{cases} a_1 = 3 & \textcircled{1} \\ a_2 = 4 & \textcircled{2} \\ a_1 + a_3 = 1 & \textcircled{3} \\ a_2 + a_3 = 2 & \textcircled{4} \end{cases}$

$\textcircled{3} - \textcircled{1}$, $\textcircled{4} - \textcircled{2}$:

$$\Rightarrow \begin{cases} a_1 = 3 & \textcircled{1} \\ a_2 = 4 & \textcircled{2} \\ a_3 = -2 & \textcircled{3} \\ a_3 = -2 & \textcircled{4} \end{cases}$$

see work on next page (P6)

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(4-3):

$$\left\{ \begin{array}{l} a_1 = 3 \text{ (1)} \\ a_2 = 4 \text{ (2)} \\ a_3 = -2 \text{ (3)} \\ 0 = 0 \text{ (4)} \end{array} \right.$$

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Then we get the solution of the system of linear equations,

$$(a_1, a_2, a_3) = (3, 4, -2)$$

$$\begin{aligned} \text{Then } P &= x^3 + 2x^2 - 3x + 4 = 3(x^3 - x) + 4(x^2 + 1) - 2(x^2 + x^3) \\ &= 3U_1 + 4U_2 - 2U_3 \end{aligned}$$

Hence P is a linear combination of U_1, U_2, U_3

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