October 1st, 2009

MAT240; Abstract Linear Algebra Lecture

Goal: Every v.s. has a basis. So while we don't *have* to work in coordinates, we always can.

$$Span(S) = \left\{ \sum_{i=1}^{n} a_i u_i \,:\, a_i \in F, u_i \in S \right\}$$

Proposition:

1. Span(S) is a subspace of V.

$$7 \sum a_i u_i = \sum (7a_i) u_i$$

$$w_i = \{u_i \;\; i \leq n, v_{i-n} \;\; i > n\}$$
$$= \sum c_i w_i$$

2. If $S_1 \subset Span(S_2)$, then $Span(S_1) \subset Span(S_2)$

$$Span(S_2) = W \subset V \rightarrow S_1 \subset W \rightarrow Span(S_1) \subset W = Span(S_2)$$

Definition:

A subset $S \subset V$ is called ("wasteful") linearly independent if $\exists \; (scalars) a_i \in F \;\&\; \exists \; distinct \; u_i \in S$ s.t.:

$a_1 u_1 + \cdots a_n u_n = \sum a_i u_i = 0$ and not all a_i's are 0.

Definition:

$S \subset V$ is linearly independent if it is not linearly dependent.

$\leftrightarrow \sum a_i u_i = 0 \rightarrow \forall i \;\; a_i = 0, \; a_i \in F \; and \; u_i \in S$

Example:

In $M_{mxn} = V$

$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is linearly independent

$$a\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + b\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + d\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

$$\begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \to a = 0, b = 0, c = 0, d = 0$$

Comments:

1. $0 = \{\quad\}$ is linearly independent
2. $\{u\}; u \in V$ is independent unless u = 0, but if u = 0 is dependent.
3. Suppose $S_1 \subset S_2 \subset V$
   a. If S_1 is linearly dependent, S_2 is linearly dependent.
   b. If S_1 is linearly independent, know not about S_2 .
   c. If S_2 is linearly independent, so is S_1.
   d. If S_2 is linearly dependent, know not about S_1.
4. If S is linearly independent and $u \in V\backslash S$ then $S \cup \{u\}$ is linearly dependent iff u is a linear combination of elements of S.

$$\leftrightarrow u \in Span(S)$$

Proof: Suppose $u \in Span(S) \to u = \sum a_i u_i \quad a_i \in F, \quad u_i \in S\ (distinct)$

$\leftrightarrow 1u - \sum a_i u_i = 0$ (non-trivial l.c. of distinct elements of $S \cup \{u\}$

$\to S \cup \{u\}$ is linearly dependent.