MAT240. Nov. 14, 2006.

Lemma. If P, Q are invertible, $\operatorname{rank} T = \operatorname{rank} PT = \operatorname{rank} TQ$

$$Q \circlearrowright V \xrightarrow{T} W \circlearrowleft P$$

Pf. $\operatorname{rank} T = \operatorname{rank} TQ$: Q is onto $\Rightarrow$ $\operatorname{range} T = \operatorname{range} TQ$
$\Rightarrow \operatorname{rank} T = \operatorname{rank} TQ$.

$\operatorname{rank} PT = \operatorname{rank} T$ in text.

Principle (to be revisited later): Changing a basis $\cong$ multiplying by an invertible matrix / linear transformation.

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \beta \to \beta' & & \gamma \to \gamma' \end{array}$$

$A = [T]_\beta^\gamma \qquad B = [T]_{\beta'}^{\gamma'} \qquad B = PAQ$ for some invertible P, Q

Cor rank A is well-defined.

Thm. Any matrix A can be row- and column- reduced to a "block" matrix of the form:

$$\left[\begin{array}{c|c} I_{r\times r} & 0 \\ \hline 0 & 0 \end{array}\right] \Rightarrow \operatorname{rank} A = r$$

Pf. If $A = 0$ then it is of that form $r = 0$.
Otherwise it has non-zero entry somewhere: By row and column swaps, bring the non-zero entry to top left.
Divide the first row by that entry, and then our matrix becomes $\begin{bmatrix} 1 & \sim \\ \wr & \ddots \end{bmatrix}$. Using row ops, kill everything on the first column other than the 1. $\begin{bmatrix} 1 & \sim\sim \\ 0 & \\ \vdots & \ddots \\ 0 & \end{bmatrix}$

Using column operations, do same to first row.

By induction, B can be reduced to $\begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & B_1 \\ 0 & \end{bmatrix}$

$B' = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

Doing the same row/column ops to the bigger matrix, we get

$$\begin{bmatrix} I & 0 \cdots 0 \\ 0 & \begin{array}{c|c} I_{p} & 0 \\ \hline 0 & 0 \end{array} \\ \vdots & \\ 0 & \end{bmatrix} = \begin{bmatrix} I_{p+1 \times p+1} & 0 \\ 0 & 0 \end{bmatrix}. \quad \square$$

Recall: If $A \in M_{m \times n}(F)$ then $A^T \in M_{n \times m}(F)$.
It is obtained by switching the rows & columns.
$(A^T)_{ij} = A_{ji}$

Claim: $\operatorname{rank} A = \operatorname{rank} A^T$

Pf.

square   rectangular

$$\begin{bmatrix} \text{dror} \\ \text{maxi} \\ \text{stan} \end{bmatrix} \xrightarrow{\text{row \& col ops}} \left[\begin{array}{c|c} I_{r \times r} & \boxed{0} \\ \hline \boxed{0} & \boxed{0} \end{array}\right] \quad \text{rank} = r$$

$$\begin{bmatrix} d & m & s \\ r & a & t \\ o & x & a \\ r & i & n \end{bmatrix} \xrightarrow{\text{col \& row ops}} \left[\begin{array}{c|c} I_{r \times r} & \boxed{0} \\ \hline \boxed{0} & \boxed{0} \end{array}\right]. \quad \text{rank} = r$$

Def Let $A \in M_{m \times n}$ matrix (F)
Column-space A = span$(c_1, \ldots c_n) \subset F^m$
row-space A = span$(r_1 \ldots r_m) \in F^n$

$$A = \underbrace{\begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix}}_{n} \Bigg\} m$$

Claim dim col-space(A) = dim row-space(A) = rank A

$c_1 \cdots c_n \in F^m$

$$A = \begin{bmatrix} - & r_1 & - \\ & \vdots & \\ - & r_m & - \end{bmatrix}$$

$r_1 \cdots r_m \in F^n$

MAT240

A defines a linear transformation $T : F^n \to F^m$
By mapping $v \longmapsto A \cdot v$

$\underset{F^n = M_{n\times 1}}{v} \longmapsto \underset{M_{m\times 1} = F^m}{A\cdot v}$ $\qquad A = \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix}$

I.e. $T(v) = A \cdot v$ $\qquad \underbrace{\begin{bmatrix} & & \\ & & \end{bmatrix}}_{n}\Big\}m \,(v) = (Av)$

It is easy to check that if (e_i) is the standard basis of $\mathbb{R}^m$ and (f_j) is the standard basis of $\mathbb{R}^n$, then

$$[T]^{(e_i)}_{(f_j)} = A \qquad \underline{\text{Pf.}}\ T(f_j) = A \cdot f_j = c_j$$

$$T \longmapsto \begin{bmatrix} & | & \\ & c_j & \\ & | & \end{bmatrix}$$

so reproduce matrix you started from

1. $\operatorname{rank} A = \operatorname{rank} T = \dim R(T) = \dim \operatorname{span}(Tf_j) = \dim \operatorname{span}(c_j)$
$= \dim \text{col-space}(A)$

This proves that $\operatorname{rank} A = \dim \text{col-space}(A)$
Yet $\operatorname{rank}(A) = \operatorname{rank}(A^T) = \dim(\text{col-space}(A^T)) = \dim(\text{row-space}(A))$.
$\square$

$\underline{\text{Q}}$. How far can you go with row-ops alone?
$\underline{\text{A}}$. You can bring A to "reduced row-echelon form".

$\underline{\text{Q}}$. Reduced row echelon form?
$\underline{\text{A}}$. It's as far as you can get with row ops.

Reduced row-echelon form

1. All zero rows are at the bottom.
2. In every non-zero row, the leading entry is 1.
3. In the column of any such 1, all other entries are zero.
4. The leading 1's are in "echelon form."

$$\begin{bmatrix} 1 & \sim & \sim \\ & 1 & \sim \\ 0 & & 1 \end{bmatrix}$$

Q. What's the rank of a matrix in reduced row echelon form?
A. It is the number of leading 1s / of non-zero rows.

Q. If A is invertible $(n\times n)$ and A' is the result of row-reducing (A' is in r.r.e.f.). What's A'?
A: $\operatorname{rank} A = n$ in A', there are n leading 1's.

$$A' = \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{n} \Big\}n \Rightarrow A' = I.$$

I.e.

$E_p \cdots E_3 E_2 E_1 A = A' = I$, where $E_1 \ldots E_p$ are elementary matrices

$$\underbrace{(E_p \cdots E_1)}_{A^{-1}}(A) = I.$$

The inverse of A is the product of elementary matrices used to row-reduce it to r.r.e.f.

$$A^{-1} = E_p \cdots E_1 I$$

↳ A^{-1} is the result of applying to I all the row ops you would have applied to A to get to r.r.e.f.

MAT240.

$$A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \xrightarrow{\text{some row ops}} A^{-1}$$

Let B be the block matrix $B = (A \mid I)$

Do row ops to B to make left half $= I$. Now read A^{-1} off the right half.